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# Stanley character formula for the spin characters of the symmetric groups

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**Abstract.** We give a new formula for the irreducible spin characters of the symmetric groups. This formula is analogous to Stanley's character formula for the usual (linear) characters of the symmetric groups.

**Keywords:** spin representations of the symmetric groups, Stanley character formula

A full version of this extended abstract [9] will be published elsewhere.

The *spin symmetric group* $\widetilde{\mathfrak{S}}_n$ is the double cover of the symmetric group $\mathfrak{S}_n$. This group is generated by $t_1, \dots, t_{n-1}, z$ subject to the relations:

$$\begin{aligned} z^2 &= 1, & & \\ zt_i &= t_i z, & t_i^2 = z \qquad & \text{for } i \in [n-1], \\ (t_i t_{i+1})^3 &= z & & \text{for } i \in [n-2], \\ t_i t_j &= z t_j t_i & & \text{for } |i-j| \geq 2; \end{aligned}$$

we use the convention that $[k] = \{1, \dots, k\}$. This group was introduced by Schur [12]; it is essential for studying *projective representations* of the usual symmetric group $\mathfrak{S}_n$.

Schur proved that, roughly speaking[1], the conjugacy classes of $\widetilde{\mathfrak{S}}_n$ which are non-trivial from the viewpoint of the character theory are indexed by *odd partitions* of $n$, i.e. partitions $(\pi_1, \dots, \pi_l)$ of $n$ such that $\pi_1 \geq \cdots \geq \pi_l$ are odd positive integers. The set of such odd partitions of $n$ will be denoted by $\mathrm{OP}_n$. We set $\mathrm{OP} = \bigcup_{n=0}^{\infty} \mathrm{OP}_n$.

The central element $z \in \widetilde{\mathfrak{S}}_n$ acts on each irreducible representation by $\pm 1$. An irreducible representation of $\widetilde{\mathfrak{S}}_n$ is said to be *spin* if $z$ corresponds to $-1$. Schur also proved

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[1] For an exact statement see [9].

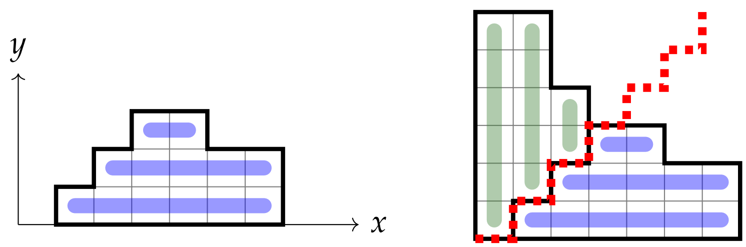


**Figure 1:** Strict partition $\xi = (6,5,2)$ shown as a *shifted Young diagram* and its double $D(\xi) = (7,7,5,3,2,2)$.

that the irreducible spin representations of $\widetilde{\mathfrak{S}}_n$, roughly speaking[2], correspond to *strict partitions* of $n$, i.e. to partitions $(\xi_1, \dots, \xi_l)$ of $n$ which form a *strictly* decreasing sequence $\xi_1 > \cdots > \xi_l$ of positive integers. The set of such strict partitions of $n$ will be denoted by $\mathrm{SP}_n$. We will represent them as *shifted Young diagrams*, cf. Figure 1. We set $\mathrm{SP} = \bigcup_{n=0}^{\infty} \mathrm{SP}_n$.

For an odd partition $\pi \in \mathrm{OP}_n$ (which corresponds to a conjugacy class of $\widetilde{\mathfrak{S}}_n$) and a strict partition $\xi \in \mathrm{SP}_n$ (which corresponds to its irreducible spin representation) we denote by $\widetilde{\phi}^{\xi}(\pi)$ the corresponding *spin character* (for some fine details related to this definition we refer to [5, Section 2] and [9]). **Our goal is to give a closed formula for such spin characters which would be useful for the purposes of the asymptotic representation theory, i.e. which would allow good understanding of the limit $\xi \to \infty$.**

In the following it will be more convenient to pass to quantities

$$X^{\xi}(\pi) := 2^{\frac{\ell(\xi)-\ell(\pi)}{2}} \, \widetilde{\phi}^{\xi}(\pi),$$

where $\ell(\pi)$ denotes the number of parts of a partition $\pi$, cf. [5, Proposition 3.3].

# 1 Normalized characters

The usual way of viewing the linear characters $\chi^{\lambda}(\pi)$ of the symmetric group $\mathfrak{S}_n$ is to fix the irreducible representation $\lambda$ and to consider the character as a function of the conjugacy class $\pi$. The *dual approach*, initiated by Kerov and Olshanski [6], suggests to do the opposite: *fix the conjugacy class $\pi$ and to view the character as a function of the irreducible representation $\lambda$*. In order for this approach to be successful one has to choose the most convenient normalization constants which we review in the following.

For a fixed integer partition $\pi$ the corresponding *normalized linear character on the*

[2]For an exact statement see [9].

*conjugacy class* $\pi$ (cf. [6]) is the function on the set of *all* Young diagrams given by

$$\operatorname{Ch}_\pi(\lambda) := \begin{cases} n^{\downarrow k} \, \frac{\chi^\lambda(\pi \cup 1^{n-k})}{\chi^\lambda(1^n)} & \text{if } n \geq k, \\ 0 & \text{otherwise,} \end{cases}$$

where $n = |\lambda|$ and $k = |\pi|$ and $n^{\downarrow k} = n(n-1)\cdots(n-k+1)$ denotes the falling power. Above, for partitions $\lambda, \sigma \vdash n$ we denote by $\chi^\lambda(\sigma)$ the irreducible linear character of the symmetric group which corresponds to the Young diagram $\lambda$, evaluated on any permutation with the cycle decomposition given by $\sigma$.

Following Ivanov [5], for a fixed odd partition $\pi \in \mathrm{OP}$ the corresponding *normalized spin character* is a function on the set of *all* strict partitions given by

$$\operatorname{Ch}^{\mathrm{spin}}_\pi(\xi) := \begin{cases} n^{\downarrow k} \, \frac{X^\xi(\pi \cup 1^{n-k})}{X^\xi(1^n)} = n^{\downarrow k} \, 2^{\frac{k-\ell(\pi)}{2}} \, \frac{\widetilde{\phi}^\xi(\pi \cup 1^{n-k})}{\widetilde{\phi}^\xi(1^n)} & \text{if } n \geq k, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

where $n = |\xi|$, $k = |\pi|$, and $\ell(\pi)$ denotes the number of parts of $\pi$. **We will find a closed formula for such spin characters $\operatorname{Ch}^{\mathrm{spin}}_\pi$. We will achieve it by finding a link between the families $(\operatorname{Ch}^{\mathrm{spin}}_\pi)$ and $(\operatorname{Ch}_\pi)$ of spin and linear characters.**

## 2 Stanley character formulas

Let $\sigma_1, \sigma_2 \in \mathfrak{S}_k$ be permutations and let $\lambda$ be a Young diagram. Following [4], we say that $(f_1, f_2)$ *is a coloring of* $(\sigma_1, \sigma_2)$ *which is compatible with* $\lambda$ if:

- $f_i \colon C(\sigma_i) \to \mathbb{Z}_+$ is a function on the set of cycles of $\sigma_i$ for each $i \in \{1, 2\}$; we view the values of $f_1$ as columns of $\lambda$ and the values of $f_2$ as rows;
- whenever $c_1 \in C(\sigma_1)$ and $c_2 \in C(\sigma_2)$ are cycles which are not disjoint, the box with the Cartesian coordinates $\big(f_1(c_1), f_2(c_2)\big)$ belongs to $\lambda$.

We denote by $N_{\sigma_1,\sigma_2}(\lambda)$ the number of colorings of $(\sigma_1, \sigma_2)$ which are compatible with $\lambda$.

*Example* 2.1. Let

$$\sigma_1 = \underbrace{(1,5,4,2)}_{V}\underbrace{(3)}_{W}, \qquad \sigma_2 = \underbrace{(2,3,5)}_{\Pi}\underbrace{(1,4)}_{\Sigma}. \tag{2.1}$$

There are three pairs of cycles $(\sigma_1, \sigma_2) \in C(\sigma_1) \times C(\sigma_2)$ with the property that $\sigma_1$ and $\sigma_2$ are not disjoint, namely $(V, \Pi), (V, \Sigma), (W, \Pi)$. It is now easy to check graphically (cf. Figure 2) that $(f_1, f_2)$ is indeed a coloring compatible with $\lambda = (3, 1)$ for

$$f_1(V) = 1, \quad f_1(W) = 3, \quad f_2(\Pi) = 1, \quad f_2(\Sigma) = 2. \tag{2.2}$$

By considering four possible choices for the values of $f_2$ and counting the choices for the values of $f_1$ one can verify that $N_{\sigma_1,\sigma_2}(\lambda) = 3^2 + 3 + 1 + 1 = 14$ for $\lambda = (3, 1)$.

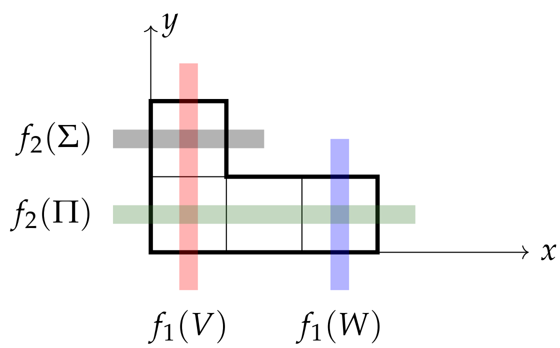


**Figure 2:** Graphical representation of the coloring (2.2) of the permutations (2.1) which is compatible with the Young diagram $\lambda = (3,1)$.

## 2.1 Linear Stanley character formula

Stanley [15] conjectured a certain closed formula for the linear characters of the symmetric groups. One of its proofs [4] was obtained by rewriting it in an equivalent form which we will recall in the following.

We will identify a given integer partition $\pi = (\pi_1, \dots, \pi_\ell) \vdash k$ with an arbitrary permutation $\pi \in \mathfrak{S}_k$ with the corresponding cycle structure.

**Theorem 2.2** ([4])**.** *For any partition $\pi \vdash k$ and any Young diagram $\lambda$*

$$\mathrm{Ch}_\pi(\lambda) = \sum_{\substack{\sigma_1,\sigma_2 \in \mathfrak{S}_k \\ \sigma_1\sigma_2=\pi}} (-1)^{\sigma_1} N_{\sigma_1,\sigma_2}(\lambda), \tag{2.3}$$

*where $(-1)^{\sigma_1} \in \{-1,1\}$ denotes the sign of the permutation $\sigma_1$.*

This formula is closely related to *Kerov polynomials* [2] which are expressions of the characters $\mathrm{Ch}_\pi$ in terms of *free cumulants* of Young diagrams. Recently, the first author [8] found spin counterparts for Kerov polynomials. The current paper was initiated by attempts to understand the underlying structures behind this result.

## 2.2 The main result: spin Stanley character formula

For a strict partition $\xi \in \mathrm{SP}_n$ we consider its *double* $D(\xi)$ which is an integer partition of $2n$. Graphically, $D(\xi)$ corresponds to a Young diagram obtained by arranging the *shifted Young diagram* $\xi$ and its 'transpose' so that they nicely fit along the 'diagonal', cf. Figure 1, see also [7, page 9].

For $\sigma_1, \sigma_2 \in \mathfrak{S}_k$ we denote by $|\sigma_1 \vee \sigma_2|$ the number of orbits in the set $[k] = \{1, \dots, k\}$ under the action of the group $\langle \sigma_1, \sigma_2 \rangle$ generated by $\sigma_1$ and $\sigma_2$. As before, we identify an integer partition $\pi \vdash k$ with an arbitrary permutation $\pi \in \mathfrak{S}_k$ with the corresponding cycle structure.

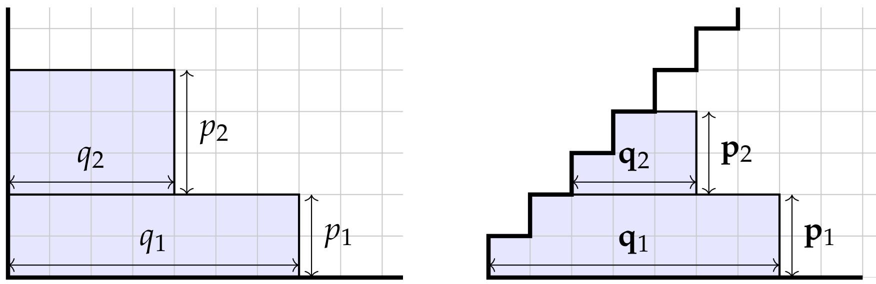


**Figure 3:** Multirectangular Young diagram $P \times Q$ and multirectangular shifted Young diagram $\mathbf{P} \rtimes \mathbf{Q}$.

**Theorem 2.3** (The main result)**.** *For any odd partition $\pi \in \mathrm{OP}_k$ and $\xi \in \mathrm{SP}$*

$$\mathrm{Ch}^{\mathrm{spin}}_{\pi}(\xi) = \sum_{\substack{\sigma_1,\sigma_2 \in \mathfrak{S}_k \\ \sigma_1\sigma_2 = \pi}} \frac{1}{2^{|\sigma_1 \vee \sigma_2|}} \, (-1)^{\sigma_1} \, N_{\sigma_1,\sigma_2}\big(D(\xi)\big). \tag{2.4}$$

The remaining sections of this paper (Sections 3 to 5) are devoted to a sketch of the proof of this result. In the following we will discuss some of its applications.

## 2.3 Application: bounds on spin characters

The following character bound is a spin version of an analogous result for the linear characters of the symmetric group [4]. It is a direct application of Theorem 2.3 and its proof follows the same line as its linear counterpart [4].

**Corollary 2.4.** *There exists a universal constant $a > 0$ with the property that for any integer $n \geq 1$, any strict partition $\xi \in \mathrm{SP}_n$, and any odd partition $\pi \in \mathrm{OP}_n$*

$$2^{\frac{n-\ell(\pi)}{2}} \left| \frac{\widetilde{\phi}^{\xi}(\pi)}{\widetilde{\phi}^{\xi}(1^n)} \right| = \left| \frac{X^{\xi}(\pi)}{X^{\xi}(1^n)} \right| < \left[ a \max\left( \frac{\xi_1}{n}, \frac{n-\ell(\pi)}{n} \right) \right]^{n-\ell(\pi)}.$$

Several asymptotic results about (random) Young diagrams and tableaux which use the inequality from [4] can be generalized in a rather straightforward way to (random) *shifted* Young diagrams and *shifted* tableaux thanks to Corollary 2.4. A good example is provided by the results about the asymptotics of the number of skew standard Young tableaux of prescribed shape [3] which can be generalized in this way to asymptotics of the number of skew *shifted* standard Young tableaux.

## 2.4 Application: characters on multirectangular Young diagrams

Following Stanley [14], for tuples of integers $P = (p_1, \ldots, p_l)$, $Q = (q_1, \ldots, q_l)$ which fulfill some obvious inequalities we consider the corresponding *multirectangular Young dia-*

*gram* $P \times Q$, cf. Figure 3. Stanley [14, 15] initiated investigation of the characters $\mathrm{Ch}_\pi(P \times Q)$ viewed as polynomials in the multirectangular coordinates $p_1, \dots, p_l, q_1, \dots, q_l$; these polynomials now are referred to as *(linear) Stanley character polynomials*.

The number of colorings $N_{\sigma_1,\sigma_2}(P \times Q) \in \mathbb{Z}[p_1, \dots, p_l, q_1, \dots, q_l]$ is given by a very explicit, convenient polynomial. In this way the linear Stanley formula (Theorem 2.2) gives an explicit expression for the linear Stanley polynomials.

De Stavola [1] adapted these concepts to shifted multirectangular Young diagrams $\mathbf{P} \rtimes \mathbf{Q}$ cf. Figure 3 and initiated investigation of *spin Stanley polynomials* $\mathrm{Ch}^{\mathrm{spin}}_\pi(\mathbf{P} \rtimes \mathbf{Q})$. Thanks to Theorem 2.3, by expressing the multirectangular coordinates $P, Q$ of the double $P \times Q = D(\mathbf{P} \rtimes \mathbf{Q})$ in terms of the shifted multirectangular coordinates $\mathbf{P}, \mathbf{Q}$ one can obtain a rather straightforward expression for the spin Stanley polynomial $\mathrm{Ch}^{\mathrm{spin}}_\pi(\mathbf{P} \rtimes \mathbf{Q})$. Applications of this result to *spin Kerov polynomials* will be discussed in a forthcoming paper.

## 2.5 Towards irreducible representations of spin groups

The proof of the linear Stanley formula (2.3) presented in [4] was found in the following way. We attempted to reverse-engineer the right-hand side of (2.3) and to find

- some natural vector space $V$ with the basis indexed by combinatorial objects; the space $V$ should be a representation of the symmetric group $\mathfrak{S}_n$ with $n := |\lambda|$, and
- a projection $\Pi \colon V \to V$ such that $\Pi$ commutes with the action of $\mathfrak{S}_n$ and such that its image $\Pi V$ is an irreducible representation of $\mathfrak{S}_n$ which corresponds to the specified Young diagram $\lambda$

in such a way that the corresponding character of $\Pi V$ would coincide with the right-hand side of (2.3).

Our attempt was successful: one could consider a vector space $V$ with the basis indexed by fillings of the boxes of $\lambda$ with the numbers from $[n]$. The action of $\mathfrak{S}_n$ on this basis was given by pointwise relabelling of the values in the boxes. The projection $\Pi$ turned out to be the Young symmetrizer with the action given by shuffling of the boxes in the rows and columns of $\lambda$. The resulting representation $\Pi V$ clearly coincides with the Specht module, which concluded the proof.

The structure of the right-hand side of (2.4) might be an indication that an analogous reverse-engineering process could be applied to the spin case. The result would be a very explicit construction of the irreducible spin representations which would be an alternative to the somewhat complicated approach of Nazarov [11].

# 3 Linear characters in terms of spin characters

For $\xi \in \mathrm{SP}$ and $\pi \in \mathrm{OP}$ we denote

$$\widetilde{\mathrm{Ch}}_\pi(\xi) := \frac{1}{2}\,\mathrm{Ch}_\pi\big(D(\xi)\big).$$

The following result is an intermediate step in the proof of Theorem 2.3 but it might be of independent interest. In particular, in a forthcoming paper [10] we shall discuss its applications in the study of random strict partitions as well as random shifted standard Young tableaux.

**Theorem 3.1.** *For any odd integers $k_1, k_2, \ldots \geq 1$ the following equalities between functions on the set* SP *of strict partitions hold true:*

$$\begin{aligned} \widetilde{\mathrm{Ch}}_{k_1} &= \mathrm{Ch}^{\mathrm{spin}}_{k_1}, \qquad (3.1)\\ \widetilde{\mathrm{Ch}}_{k_1,k_2} &= \mathrm{Ch}^{\mathrm{spin}}_{k_1,k_2} + \mathrm{Ch}^{\mathrm{spin}}_{k_1}\,\mathrm{Ch}^{\mathrm{spin}}_{k_2},\\ \widetilde{\mathrm{Ch}}_{k_1,k_2,k_3} &= \mathrm{Ch}^{\mathrm{spin}}_{k_1,k_2,k_3} + \mathrm{Ch}^{\mathrm{spin}}_{k_1,k_2}\,\mathrm{Ch}^{\mathrm{spin}}_{k_3} + \mathrm{Ch}^{\mathrm{spin}}_{k_1,k_3}\,\mathrm{Ch}^{\mathrm{spin}}_{k_2} + \mathrm{Ch}^{\mathrm{spin}}_{k_2,k_3}\,\mathrm{Ch}^{\mathrm{spin}}_{k_1},\\ &\vdots\\ \widetilde{\mathrm{Ch}}_{k_1,\ldots,k_l} &= \sum_{\substack{I:\\ |I|\leq 2}} \prod_{b\in I} \mathrm{Ch}^{\mathrm{spin}}_{(k_i:i\in b)}, \qquad (3.2) \end{aligned}$$

*where the sum in (3.2) runs over all set-partitions of $[l]$ into at most two blocks.*

*Proof.* For an integer partition $\pi$ we consider the standard numerical factor

$$z_\pi = \prod_{j\geq 1} j^{m_j(\pi)} m_j(\pi)!,$$

where $m_j(\pi)$ denotes the *multiplicity* of $j$ in the partition $\pi$. We denote by $f^\lambda = \chi^\lambda\left(1^{|\lambda|}\right)$ the number of standard tableaux of shape $\lambda$. For a strict partition $\xi$ we denote

$$g^\xi = X^\xi\left(1^{|\xi|}\right)$$

which also happens to be the number of *shifted* standard tableaux with the shape given by the shifted Young diagram $\xi$, see [7, III–8, Ex. 12].

Recall the symmetric function algebra $\Lambda = \mathbb{C}[p_1, p_2, p_3, \ldots]$ and its subalgebra, *the algebra of supersymmetric functions* $\Gamma = \mathbb{C}[p_1, p_3, p_5, \ldots]$, where the $p_r$ are Newton's power-sums. Define the algebra homomorphism $\varphi : \Lambda \to \Gamma$ by

$$\varphi(p_r) = \begin{cases} 2p_r & \text{if } r \text{ is odd,}\\ 0 & \text{if } r \text{ is even.}\end{cases}$$

Then [7, III–8, Ex. 10] implies that for any strict partition $\xi$ we have

$$\varphi(s_{D(\xi)}) = 2^{-\ell(\xi)}(Q_\xi)^2, \tag{3.3}$$

where $Q_\xi = Q_\xi(x;-1)$ denotes Schur's $Q$-function [7, pp. III–8].

Recall the Frobenius formula for Schur functions:

$$s_\mu = \sum_\pi z_\pi^{-1}\, \chi^\mu(\pi)\; p_\pi.$$

Applying the homomorphism $\varphi$ to this identity with $\mu = D(\xi)$, we obtain

$$\varphi(s_{D(\xi)}) = \sum_{\pi \in \mathrm{OP}_{2n}} 2^{\ell(\pi)} z_\pi^{-1}\, \chi^{D(\xi)}(\pi)\; p_\pi.$$

And, recall the Frobenius formula for Schur $Q$-functions:

$$Q_\xi = \sum_{\nu \in \mathrm{OP}_n} 2^{\ell(\nu)} z_\nu^{-1}\, X^\xi(\nu)\; p_\nu.$$

Substituting these formulas to (3.3), we have for any $\xi \in \mathrm{SP}_n$

$$\sum_{\pi \in \mathrm{OP}_{2n}} 2^{\ell(\pi)} z_\pi^{-1}\, \chi^{D(\xi)}(\pi)\; p_\pi = 2^{-\ell(\xi)} \left( \sum_{\nu \in \mathrm{OP}_n} 2^{\ell(\nu)} z_\nu^{-1}\, X^\xi(\nu)\; p_\nu \right)^2. \tag{3.4}$$

By comparing the coefficients of $p_{(1^{2n})} = p_{(1^n)} p_{(1^n)}$ in both sides of (3.4), we find

$$\frac{f^{D(\xi)}}{(2n)!} = 2^{-\ell(\xi)} \left( \frac{g^\xi}{n!} \right)^2. \tag{3.5}$$

First we assume that $\pi$ is an odd partition which does not have parts equal to 1, i.e., $m_1(\pi) = 0$. By comparing the coefficients of $p_{\pi \cup (1^{2n-|\pi|})}$ in both sides of (3.4) we find

$$\frac{\chi^{D(\xi)}\left(\pi \cup (1^{n-|\pi|})\right)}{z_{\pi \cup (1^{n-|\pi|})}} = 2^{-\ell(\xi)} \sum_{\substack{\mu^1,\mu^2 \\ \mu^1 \cup \mu^2 = \pi}} \frac{X^\xi\left(\mu^1 \cup (1^{n-|\mu^1|})\right)}{z_{\mu^1 \cup (1^{n-|\mu^1|})}} \frac{X^\xi\left(\mu^2 \cup (1^{n-|\mu^2|})\right)}{z_{\mu^1 \cup (1^{n-|\mu^2|})}}.$$

By the assumption $m_1(\pi) = 0$, we have $z_{\pi \cup (1^{2n-|\pi|})} = z_\pi \cdot (2n - |\pi|)!$ and $z_{\mu^i \cup (1^{n-|\mu^i|})} = z_{\mu^i} \cdot (n - |\mu^i|)!$. Thus, we obtain

$$\frac{\chi^{D(\xi)}\left(\pi \cup (1^{n-|\pi|})\right)}{z_\pi \cdot (2n - |\pi|)!} = 2^{-\ell(\xi)} \sum_{\substack{\mu^1,\mu^2 \\ \mu^1 \cup \mu^2 = \pi}} \frac{X^\xi\left(\mu^1 \cup (1^{n-|\mu^1|})\right)}{z_{\mu^1} \cdot (n - |\mu^1|)!} \frac{X^\xi\left(\mu^2 \cup (1^{n-|\mu^2|})\right)}{z_{\mu^2} \cdot (n - |\mu^2|)!}.$$

Taking the quotient of this and (3.5), we have

$$\frac{1}{z_\pi}\frac{(2n)!}{(2n-|\pi|)!}\frac{\chi^{D(\xi)}\left(\pi\cup(1^{2n-|\pi|})\right)}{f^{D(\xi)}} = \sum_{\substack{\mu^1,\mu^2\\ \mu^1\cup\mu^2=\pi}} \frac{1}{z_{\mu^1}z_{\mu^2}}\frac{n!}{(n-|\mu^1|)!}\frac{X^\xi\left(\mu^1\cup(1^{n-|\mu^1|})\right)}{g^\xi}\frac{n!}{(n-|\mu^2|)!}\frac{X^\xi\left(\mu^2\cup(1^{n-|\mu^2|})\right)}{g^\xi},$$

which is equivalent to

$$\mathrm{Ch}_\pi\big(D(\xi)\big) = \sum_{\substack{\mu^1,\mu^2\\ \mu^1\cup\mu^2=\pi}} \frac{z_\pi}{z_{\mu^1}z_{\mu^2}}\mathrm{Ch}^{\mathrm{spin}}_{\mu^1}(\xi)\,\mathrm{Ch}^{\mathrm{spin}}_{\mu^2}(\xi).$$

It is easy to see that this is equivalent to the desired formula. Thus, we completed the proof of the theorem under the assumption $m_1(\pi)=0$.

In the general case we write $\pi=\tilde{\pi}\cup(1^r)$ with $m_1(\tilde{\pi})=0$ and $r=m_1(\pi)$. We apply Theorem 3.1 for $\tilde{\pi}$; simple manipulations with the binomial coefficients imply that the claim holds true for $\pi$ as well. □

# 4 Spin characters in terms of linear characters

Formulas (3.1)–(3.2) can be viewed as an upper-triangular system of equations with unknowns $(\mathrm{Ch}^{\mathrm{spin}}_\pi)_{\pi\in\mathrm{OP}}$. It can be solved, for example

$$\left.\begin{aligned}
\mathrm{Ch}^{\mathrm{spin}}_{k_1} &= \widetilde{\mathrm{Ch}}_{k_1},\\
\mathrm{Ch}^{\mathrm{spin}}_{k_1,k_2} &= \widetilde{\mathrm{Ch}}_{k_1,k_2} - \widetilde{\mathrm{Ch}}_{k_1}\,\widetilde{\mathrm{Ch}}_{k_2},\\
\mathrm{Ch}^{\mathrm{spin}}_{k_1,k_2,k_3} &= \widetilde{\mathrm{Ch}}_{k_1,k_2,k_3}\\
&\quad - \widetilde{\mathrm{Ch}}_{k_1,k_2}\,\widetilde{\mathrm{Ch}}_{k_3} - \widetilde{\mathrm{Ch}}_{k_1,k_3}\,\widetilde{\mathrm{Ch}}_{k_2} - \widetilde{\mathrm{Ch}}_{k_2,k_3}\,\widetilde{\mathrm{Ch}}_{k_1}\\
&\quad + 3\widetilde{\mathrm{Ch}}_{k_1}\widetilde{\mathrm{Ch}}_{k_2}\widetilde{\mathrm{Ch}}_{k_3},\\
&\ \ \vdots
\end{aligned}\right\} \tag{4.1}$$

The general pattern is given by the following result. In this way several problems involving spin characters are reduced to investigation of their linear counterparts.

**Theorem 4.1.** *For any* $\pi\in\mathrm{OP}$

$$\mathrm{Ch}^{\mathrm{spin}}_\pi = \sum_I (-1)^{|I|-1}\,(2|I|-3)!!\,\prod_{b\in I}\widetilde{\mathrm{Ch}}_{(\pi_i:i\in b)}, \tag{4.2}$$

*where the sum runs over all set-partitions of the set* $[\ell(\pi)]$; *by definition* $(-1)!!=1$.

*Proof.* The process of solving the upper-triangular system of equations (3.1)–(3.2) can be formalized as follows. By singling out the partition $I$ in (3.2) which consists of exactly one block we may express the spin character $\operatorname{Ch}^{\mathrm{spin}}_{\pi}$ in terms of the linear character $\widetilde{\operatorname{Ch}}_{\pi}$ and spin characters $\operatorname{Ch}^{\mathrm{spin}}_{\pi'}$ which correspond to partitions $\pi' \in \mathrm{OP}$ with $\ell(\pi') < \ell(\pi)$:

$$\operatorname{Ch}^{\mathrm{spin}}_{\pi} = \widetilde{\operatorname{Ch}}_{\pi} - \sum_{\substack{I: \\ |I|=2}} \prod_{b\in I} \operatorname{Ch}^{\mathrm{spin}}_{(\pi_i : i\in b)}. \tag{4.3}$$

By applying this procedure recursively to the spin characters on the right-hand side, we end up with an expression for $\operatorname{Ch}^{\mathrm{spin}}_{\pi}$ as a linear combination (with integer coefficients) of the products of the form

$$\prod_{b\in I} \widetilde{\operatorname{Ch}}_{(\pi_i : i\in b)} \tag{4.4}$$

over set-partitions $I$ of $[\ell(\pi)]$. The remaining difficulty is to determine the exact value of the coefficient of (4.4) in this linear combination.

The above recursive procedure can be encoded by a tree in which each non-leaf vertex has two children and the leaves are labelled by the factors in (4.4) or, equivalently, by the blocks of the set-partition $I$. Such trees are known under the name of *total binary partitions*; the cardinality of such trees with prescribed leaf labels $I$ is equal to $(2|I|-3)!!$ [13, Example 5.2.6].

Our recursive procedure involves change of the sign; such a change occurs once for each non-leaf vertex. Thus each total binary tree contributes with multiplicity $(-1)^{|I|-1}$ which concludes the proof. □

# 5 Proof of spin Stanley formula

*Proof of Theorem 2.3.* We start with Theorem 4.1 and substitute each normalized linear character $\operatorname{Ch}_{\nu}$ which contributes to the right-hand side of (4.2) by linear Stanley character formula (2.3).

We shall discuss in detail the case when $\pi = (\pi_1, \pi_2)$ consists of just two parts. We will view $\mathfrak{S}_{\pi_1}$, $\mathfrak{S}_{\pi_2}$ and $\mathfrak{S}_{\pi_1+\pi_2}$ as the groups of permutations of, respectively, the set $\{1, \dots, \pi_1\}$, $\{\pi_1+1, \dots, \pi_1+\pi_2\}$ and $\{1, \dots, \pi_1+\pi_2\}$. In this way we may identify $\mathfrak{S}_{\pi_1} \times \mathfrak{S}_{\pi_2}$ as a subgroup of $\mathfrak{S}_{\pi_1+\pi_2}$. Thanks to these notations

$$\operatorname{Ch}^{\mathrm{spin}}_{\pi_1,\pi_2}(\xi) = \frac{(-1)!!}{2} \operatorname{Ch}_{\pi_1,\pi_2}\big(D(\xi)\big) - \frac{1!!}{2^2} \operatorname{Ch}_{\pi_1}\big(D(\xi)\big) \operatorname{Ch}_{\pi_2}\big(D(\xi)\big) =$$

$$\frac{(-1)!!}{2} \sum_{\substack{\sigma_1,\sigma_2\in\mathfrak{S}_{\pi_1+\pi_2} \\ \sigma_1\sigma_2=(\pi_1,\pi_2)}} (-1)^{\sigma_1} N_{\sigma_1,\sigma_2}\big(D(\xi)\big) - \frac{1!!}{2^2} \sum_{\substack{\sigma_1,\sigma_2\in\mathfrak{S}_{\pi_1}\times\mathfrak{S}_{\pi_2} \\ \sigma_1\sigma_2=(\pi_1,\pi_2)}} (-1)^{\sigma_1} N_{\sigma_1,\sigma_2}\big(D(\xi)\big), \tag{5.1}$$

where the last equality follows from the observation that a double sum over factorizations of $\pi_1 \in \mathfrak{S}_{\pi_1}$ and over factorizations of $\pi_2 \in \mathfrak{S}_{\pi_2}$ can be combined into a single sum over factorizations of $(\pi_1, \pi_2) \in \mathfrak{S}_{\pi_1} \times \mathfrak{S}_{\pi_2}$.

In general,

$$\mathrm{Ch}_{\pi}^{\mathrm{spin}}(\xi) = \sum_{\substack{\sigma_1,\sigma_2 \in \mathfrak{S}_{|\pi|} \\ \sigma_1\sigma_2 = \pi}} c_{\sigma_1,\sigma_2} \, (-1)^{\sigma_1} \, N_{\sigma_1,\sigma_2}\big(D(\xi)\big) \tag{5.2}$$

for some combinatorial factor $c_{\sigma_1,\sigma_2}$. The exact value of this factor is equal to

$$c_{\sigma_1,\sigma_2} = C_m = (-1) \sum_p \left\{ {m \atop p} \right\} \left(-\frac{1}{2}\right)^p (2p-3)!!, \tag{5.3}$$

where $m$ denotes the number of orbits in $[|\pi|]$ under the action of $\langle \sigma_1, \sigma_2 \rangle$, and $\left\{ {m \atop p} \right\}$ denotes Stirling numbers of the second kind. Indeed, the set-partition $I$ (over which we sum in (4.2)) can be identified with a set-partition of the set $C(\pi)$ of the cycles of the permutation $\pi \in \mathfrak{S}_{|\pi|}$. With this in mind we see that to $c_{\sigma_1,\sigma_2}$ contribute only these set-partitions $I$ on the right-hand side of (4.2) for which $I$ is bigger than the set-partition given by the orbits of $\langle \sigma_1, \sigma_2 \rangle$. The collection of such set-partitions can be identified with the collection of set-partitions of an $m$-element set (i.e. the set of orbits of $\langle \sigma_1, \sigma_2 \rangle$).

The exact form of the right-hand side of (5.3) is not important; the key point is that it depends only on $m$, the number of orbits of $\langle \sigma_1, \sigma_2 \rangle$. In order to evaluate its exact value $C_m$ we shall consider (5.2) in the special case of $\pi = 1^m$. In this case $\sigma_2 = \sigma_1^{-1}$; we denote by $l = |C(\sigma_1)|$ the number of cycles of $\sigma_1$. It follows that

$$\mathrm{Ch}_{1^m}^{\mathrm{spin}}(\xi) = n^{\downarrow m} = \sum_l \begin{bmatrix} m \\ l \end{bmatrix} C_l \, (-1)^{m-l} \, (2n)^l,$$

where $n = |\xi|$ and $\left[ {m \atop l} \right]$ denotes Stirling number of the first kind. Both sides of the equality are polynomials in the variable $n$; by comparing the leading coefficients we conclude that

$$C_m = \frac{1}{2^m}.$$

By substituting this value to (5.2) we conclude the proof. □

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