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Cyclic quasi-symmetric functions

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Abstract. The ring of cyclic quasi-symmetric functions is introduced in this paper. A natural basis consists of fundamental cyclic quasi-symmetric functions; they arise as toric P-partition enumerators, for toric posets P with a total cyclic order. The associated structure constants are determined by cyclic shuffles of permutations. For every non-hook shape λ, the coefficients in the expansion of the Schur function s_λ in terms of fundamental cyclic quasi-symmetric functions are nonnegative. The theory has applications to the enumeration of cyclic shuffles and SYT by cyclic descents.

1 Introduction

The graded rings Sym and QSym, of symmetric and quasi-symmetric functions, respectively, have many applications to enumerative combinatorics, as well as to other branches of mathematics; see, e.g., [11, Ch. 7]. This paper introduces two intermediate objects: the graded ring cQSym of cyclic quasi-symmetric functions, and its subring cQSym^-.

The rings Sym, QSym and cQSym may be defined via invariance properties. A formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree is *symmetric* if for any $t \geq 1$, any two sequences $i_1, \ldots, i_t$ and $j_1, \ldots, j_t$ of distinct positive integers (indices), and any sequence $m_1, \ldots, m_t$ of positive integers (exponents), the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{j_1}^{m_1} \cdots x_{j_t}^{m_t}$ in f are equal. We call f *quasi-symmetric* if for any $t \geq 1$, any two *increasing* sequences $i_1 < \cdots < i_t$ and $j_1 < \cdots < j_t$ of positive integers, and any sequence $m_1, \ldots, m_t$ of positive integers, the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{j_1}^{m_1} \cdots x_{j_t}^{m_t}$ in f are equal.

Definition 1.1. A *cyclic quasi-symmetric function* is a formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree such that, for any $t \geq 1$, any two increasing sequences $i_1 < \cdots < i_t$ and $i'_1 < \cdots < i'_t$ of positive integers, any sequence $m = (m_1, \ldots, m_t)$ of positive integers,

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and any *cyclic shift* $m' = (m'_1, \ldots, m'_t)$ of m, the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{i'_1}^{m'_1} \cdots x_{i'_t}^{m'_t}$ in f are equal.

Denote by cQSym the set of all cyclic quasi-symmetric functions, and by cQSym_n the set of all cyclic quasi-symmetric functions which are homogeneous of degree n. It will be shown that cQSym is a graded ring; see Proposition 3.18.

Toric posets were recently introduced by Develin, Macauley and Reiner [4]. A toric analogue of P-partitions is presented in Section 3.1. Toric P-partition enumerators, in the special case of total cyclic orders, form a convenient $\mathbb{Q}$-basis for a ring cQSym^-, which is a subring of cQSym. A slightly extended set actually forms a $\mathbb{Q}$-basis for cQSym itself. The elements of this basis are called *fundamental cyclic quasi-symmetric functions*, are indexed by cyclic compositions of a positive integer n (equivalently, by cyclic equivalence classes of nonempty subsets $J \subseteq [n]$), and are denoted $F^{\mathrm{cyc}}_{n,[J]}$. Normalized versions of them actually form $\mathbb{Z}$-bases for cQSym and cQSym^-; see Proposition 2.4.

A toric analogue of Stanley's fundamental decomposition lemma for P-partitions [12, Lemma 3.15.3], given in Lemma 3.11 below, is applied to provide a combinatorial interpretation of the resulting structure constants in terms of shuffles of cyclic permutations (more accurately, cyclic words), as follows.

For a finite set A of size a, let $\mathfrak{S}_A$ be the set of all bijections $u\colon [a] \to A$, viewed as words $u = (u_1, \ldots, u_a)$. Elements of $\mathfrak{S}_A$ will be called *bijective words*, or simply *words*. If A is a finite set of integers, or any finite totally ordered set, define the *cyclic descent set* of $u \in \mathfrak{S}_A$ by

$$\mathrm{cDes}(u) := \{1 \le i \le a \,:\, u_i > u_{i+1}\} \subseteq [a], \tag{1.1}$$

with the convention $u_{a+1} := u_1$. The *cyclic descent number* of u is $\mathrm{cdes}(u) := |\mathrm{cDes}(u)|$. A *cyclic word* $[\vec{u}] \in \mathfrak{S}_A/\mathbb{Z}_a$ is an equivalence class of elements of $\mathfrak{S}_A$ under the cyclic equivalence relation $(u_1, \ldots, u_a) \sim (u_{i+1} \ldots, u_a, u_1, \ldots, u_i)$ for all i. A *cyclic shuffle* of two cyclic words $[\vec{u}]$ and $[\vec{v}]$ with disjoint supports is the cyclic equivalence class $[\vec{w}]$ represented by any shuffle w of a representative of $[\vec{u}]$ and a representative of $[\vec{v}]$. The set of all cyclic shuffles of $[\vec{u}]$ and $[\vec{v}]$ is denoted $[\vec{u}] \sqcup\!\sqcup_{\mathrm{cyc}} [\vec{v}]$, and is clearly a union of cyclic equivalence classes.

The following cyclic analogue of Stanley's shuffling theorem [11, Ex. 7.93] provides a combinatorial interpretation for the structure constants of cQSym^-.

Theorem 1.2. *Let $C = A \sqcup B$ be a disjoint union of finite sets of integers. For each $u \in \mathfrak{S}_A$ and $v \in \mathfrak{S}_B$, one has the following expansion:*

$$F^{\mathrm{cyc}}_{|A|,[\mathrm{cDes}(u)]} \cdot F^{\mathrm{cyc}}_{|B|,[\mathrm{cDes}(v)]} = \sum_{[\vec{w}] \,\in\, [\vec{u}] \sqcup\!\sqcup_{\mathrm{cyc}} [\vec{v}]} F^{\mathrm{cyc}}_{|C|,[\mathrm{cDes}(w)]}.$$

Recall that a skew shape is called a *ribbon* if it does not contain a 2×2 square.

Theorem 1.3. *For every skew shape λ/μ which is not a connected ribbon, all the coefficients in the expansion of the skew Schur function $s_{\lambda/\mu}$ in terms of normalized fundamental cyclic quasi-symmetric functions are nonnegative integers.*

A more precise statement, which provides a combinatorial interpretation of the coefficients, is given in Theorem 4.4 below. The proof relies on the existence of a cyclic extension of the descent map on standard Young tableaux (SYT) of shape λ/μ, which was proved in [2]. Using Postnikov's result regarding toric Schur functions, one deduces that the coefficients in the expansion of a non-hook Schur function s_λ in terms of fundamental cyclic quasi-symmetric functions are equal to certain Gromov-Witten invariants.

Applications to the enumeration of SYT and cyclic shuffles of permutations with prescribed cyclic descent set or number follow from this theory. Using a ring homomorphism from cQSym to the ring of formal power series $\mathbb{Z}[[q]]_\odot$, with product defined by $q^i \odot q^j := q^{\max(i,j)}$, Theorem 1.2 implies the following result.

Theorem 1.4. *Let A and B be two disjoint sets of integers with $|A| = m$ and $|B| = n$. For each $u \in \mathfrak{S}_A$ and $v \in \mathfrak{S}_B$ the following holds.*

1. *If $\operatorname{des}(u) = i$ and $\operatorname{des}(v) = j$ then the number of shuffles of u and v with descent number k is equal to*
$$\binom{m+j-i}{k-i}\binom{n+i-j}{k-j}.$$
2. *If $\operatorname{cdes}(u) = i$ and $\operatorname{cdes}(v) = j$ then the number of cyclic shuffles of $[\vec{u}]$ and $[\vec{v}]$ with cyclic descent number k is equal to*
$$\frac{k(m-i)(n-j)+(m+n-k)ij}{(m+j-i)(n+i-j)}\binom{m+j-i}{k-i}\binom{n+i-j}{k-j}.$$

The group ring $\mathbb{Z}[\mathfrak{S}_n]$ has a distinguished subring, *Solomon's descent algebra* $\mathfrak{D}_n$, with basis elements
$$D_I := \sum_{\substack{\pi\in\mathfrak{S}_n \\ \operatorname{Des}(\pi)=I}} \pi \qquad (I \subseteq [n-1]).$$
Cellini [3] and others looked for an appropriate *cyclic* analogue. We provide a partial answer, using an operation dual to the product in $\mathfrak{D}_n$ — the *internal coproduct* Δ_n on QSym_n.

Theorem 1.5. cQSym_n *and* cQSym_n^- *are right coideals of* QSym_n *with respect to the internal coproduct:*
$$\Delta_n(\mathrm{cQSym}_n) \subseteq \mathrm{cQSym}_n \otimes \mathrm{QSym}_n$$
and
$$\Delta_n(\mathrm{cQSym}_n^-) \subseteq \mathrm{cQSym}_n^- \otimes \mathrm{QSym}_n\,.$$
The structure constants for cQSym_n^- *are nonnegative integers.*

Corollary 1.6. *For $n > 1$ let $c2^{[n]}_{0,n}$ be the set of equivalence classes, under cyclic rotations, of subsets $\varnothing \subsetneq J \subsetneq [n]$. Defining*

$$cD_A := \sum_{\substack{\pi \in \mathfrak{S}_n \\ \mathrm{cDes}(\pi) \in A}} \pi \qquad (A \in c2^{[n]}_{0,n}),$$

the additive free abelian group

$$\mathfrak{cD}_n := \mathrm{span}_{\mathbb{Z}}\{cD_A \,:\, A \in c2^{[n]}_{0,n}\}$$

is a left module for Solomon's descent algebra $\mathfrak{D}_n$.

This is an extended abstract. Proofs and more details are given in the full version of the paper [1].

2 The fundamental cyclic quasi-symmetric functions

Definition 2.1. For $n \geq 1$ and a subset $J \subseteq [n]$, denote by $P^{\mathrm{cyc}}_{n,J}$ the set of all pairs (w, k) consisting of a word $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ and an index $k \in [n]$ satisfying

(i) $w_k \leq w_{k+1} \leq \ldots \leq w_n \leq w_1 \leq \ldots \leq w_{k-1}$.

(ii) If $j \in J \setminus \{k-1\}$ then $w_j < w_{j+1}$, where indices are computed modulo n.

Example 2.2. Let $n = 5$ and $J = \{1, 3\}$. The pairs $(12345, 1)$, $(23312, 4)$ and $(23122, 3)$ are in $P^{\mathrm{cyc}}_{5,\{1,3\}}$ (see Figure 1), but the pairs $(12354, 1)$, $(22312, 4)$ and $(23112, 3)$ are not.

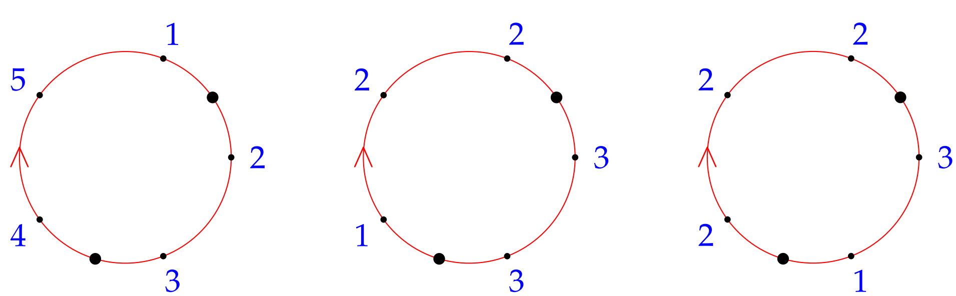


Figure 1: The pairs $(12345, 1)$, $(23312, 4)$ and $(23122, 3)$ in $P^{\mathrm{cyc}}_{5,\{1,3\}}$.

Definition 2.3. Let $c2^{[n]}$ be the set of equivalence classes, under cyclic rotations, of subsets $\varnothing \subseteq J \subseteq [n]$. For any subset $J \subseteq [n]$ and orbit $A \in c2^{[n]}$ define the *fundamental cyclic quasi-symmetric function* corresponding to J or A by

$$F_{n,J}^{\text{cyc}} := \sum_{(w,k) \in P_{n,J}^{\text{cyc}}} x_{w_1} x_{w_2} \cdots x_{w_n} \quad \text{and} \quad F_{n,A}^{\text{cyc}} := F_{n,J}^{\text{cyc}} \quad (\forall J \in A).$$

The corresponding *normalized fundamental cyclic quasi-symmetric function* is

$$\widehat{F}_{n,A}^{\text{cyc}} := \frac{1}{n} \sum_{J \in A} F_{n,J}^{\text{cyc}}.$$

It is shown that these are all well-defined (i.e., independent of the choice of $J \in A$).

Proposition 2.4. *For each $n \geq 1$, the set $\left\{ \widehat{F}_{n,A}^{\text{cyc}} \,:\, A \in c2^{[n]} \setminus \{[\varnothing]\} \right\}$ is a $\mathbb{Z}$-basis for* cQSym_n.

For many combinatorial applications it is natural to consider a certain subring cQSym_n^- of cQSym_n. Define

$$\text{cQSym}_n^- := \text{span}_{\mathbb{Z}} \left\{ \widehat{F}_{n,A}^{\text{cyc}} \,:\, A \in c2^{[n]} \setminus \{[\varnothing], [[n]]\} \right\} \qquad (n > 1),$$

as well as $\text{cQSym}_1^- := \text{span}_{\mathbb{Z}} \left\{ \widehat{F}_{1,[[1]]}^{\text{cyc}} \right\}$, $\text{cQSym}_0^- := \mathbb{Z}$, and $\text{cQSym}^- := \bigoplus_{n \geq 0} \text{cQSym}_n^-$.

3 Toric posets and cyclic *P*-partitions

We recall *toric posets* from [4], and develop for them a theory of cyclic P-partitions. In particular, we provide a cyclic analogue of Stanley's fundamental decomposition lemma for P-partitions. Just as fundamental quasi-symmetric functions $F_{n,J}$ are P-partition enumerators for certain (labeled) total orders, the fundamental cyclic quasi-symmetric functions $F_{n,J}^{\text{cyc}}$ are cyclic P-partition enumerators for certain (labeled) total cyclic orders. This is used to prove that cQSym^- is a ring and to study its structure constants.

3.1 Toric DAGs, toric posets, and toric *P*-partitions

In this section, $\vec{D}$ denotes a directed acyclic graph (DAG) with vertex set $[n] := \{1, 2, \ldots, n\}$. Usual P-partitions use posets instead of DAGs, but the toric analogue will require DAGs.

A $\vec{D}$*-partition* is a function $f \colon \{1, 2, \ldots, n\} \to \{0, 1, 2, \ldots\}$ for which

- $f(i) \leq f(j)$ whenever $i \to j$ in $\vec{D}$, and
- $f(i) < f(j)$ whenever $i \to j$ in $\vec{D}$ but $i >_{\mathbb{Z}} j$.

Denote by $\mathcal{A}(\vec{D})$ the set of all $\vec{D}$-partitions f.

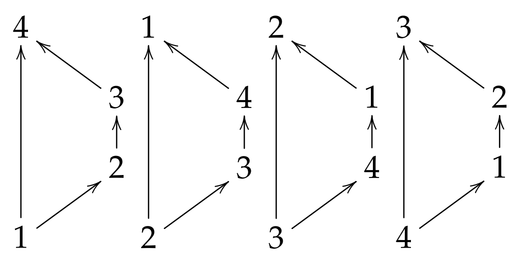
Lemma 3.1. *(Fundamental lemma of $\vec{D}$-partitions [12, Lemma 3.15.3]) For any DAG $\vec{D}$, one has a decomposition of $\mathcal{A}(\vec{D})$ as the following disjoint union:*

$$\mathcal{A}(\vec{D}) = \bigsqcup_{w \in \mathcal{L}(\vec{D})} \mathcal{A}(\vec{w}),$$

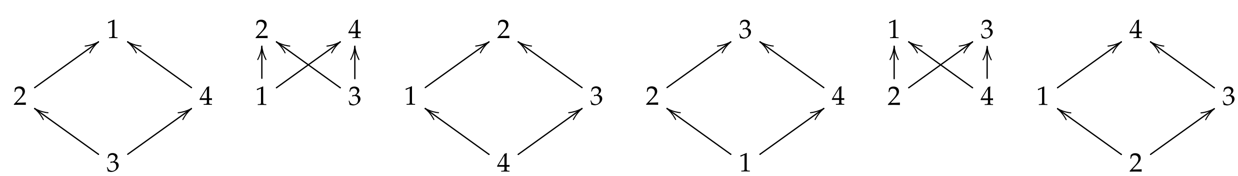
where $\mathcal{L}(\vec{D})$ is the set of all linear (total) orders which extend $\vec{D}$.

Definition 3.2. (i) $i_0 \in [n]$ is a *source* (respectively, *sink*) in $\vec{D}$ if $\vec{D}$ contains no arrows of the form $j \to i_0$ (respectively, of the form $i_0 \to j$).
(ii) $\vec{D}'$ is obtained from $\vec{D}$ by a *flip at* i_0 if i_0 is either a source or a sink of $\vec{D}$ and one obtains $\vec{D}'$ by reversing all the arrows in $\vec{D}$ incident with i_0.
(iii) Define the equivalence relation $\equiv$ on DAGs to be the reflexive-transitive closure of the flips, that is, $\vec{D} \equiv \vec{D}'$ if and only if there exists a (possibly empty) sequence of flips one can apply starting with $\vec{D}$ to obtain $\vec{D}'$.
(iv) A *toric DAG* is the $\equiv$-equivalence class $[\vec{D}]$ of a DAG $\vec{D}$.

Example 3.3. Here is an example of a toric DAG $[\vec{D}_1]$:



Here is another toric DAG $[\vec{D}_2]$:



Definition 3.4. Say that $[\vec{D}_2]$ *torically extends* $[\vec{D}_1]$ if there exist $\vec{D}'_i \in [\vec{D}_i]$ for $i = 1, 2$ with $\vec{D}'_1 \subseteq \vec{D}'_2$.

A certain toric extension, called the toric transitive closure, will be particularly important.

Definition 3.5. (i) Say that $i \to j$ is implied from *toric transitivity* in a DAG $\vec{D}$ if there exist in $\vec{D}$ both a chain $i_1 \to i_2 \to \cdots \to i_k$ and a direct arrow $i_1 \to i_k$ such that $i = i_a, j = i_b$ for some $1 \le a < b \le k$.
(ii) The *toric transitive closure* of $\vec{D}$ is the DAG $\vec{P}$ obtained by adding in all arrows $i \to j$ implied from toric transitivity in $\vec{D}$.
(iii) A DAG $\vec{D}$ is *toric transitively closed* if it equals its toric transitive closure.

Proposition 3.6. *If $\vec{D}_1 \equiv \vec{D}_2$, then $\vec{D}_1$ is toric transitively closed if and only if so is $\vec{D}_2$.*

Definition 3.7. A toric DAG $[\vec{D}]$ is a *toric poset* if $\vec{D}$ is toric transitively closed for one of its $\equiv$-class representatives $\vec{D}$, or equivalently, by Proposition 3.6, for *all* such $\vec{D}$.

Definition 3.8. A *total cyclic order* is a toric poset with at least one (equivalently, all) of its $\equiv$-class representatives being a total (linear) order.

Denote by $\mathcal{L}^{\text{tor}}([\vec{D}])$ the set of all total cyclic orders $[\vec{w}]$ which torically extend $[\vec{D}]$.

Remark 3.9. Total cyclic orders may be geometrically visualized as n dots in a directed cycle labeled by $1, \ldots, n$ with no repeats. These configurations are called *cyclic permutations*, and will be used in the study of cyclic shuffles, see Figure 2.

Definition 3.10. A *toric* $[\vec{D}]$*-partition* is a function $f \colon \{1, 2, \ldots, n\} \to \{0, 1, 2, \ldots\}$ which is a $\vec{D}'$-partition for at least one DAG $\vec{D}'$ in $[\vec{D}]$. Let $\mathcal{A}^{\text{tor}}([\vec{D}])$ denote the set of all toric $[\vec{D}]$-partitions

Lemma 3.11. *(Fundamental lemma of toric $\vec{D}$-partitions) For any DAG $\vec{D}$, one has a decomposition of $\mathcal{A}^{\text{tor}}([\vec{D}])$ as the following disjoint union:*

$$\mathcal{A}^{\text{tor}}([\vec{D}]) = \bigsqcup_{[\vec{w}] \in \mathcal{L}^{\text{tor}}([\vec{D}])} \mathcal{A}^{\text{tor}}([\vec{w}]).$$

3.2 Cyclic P-partition enumerators

Definition 3.12. Given a toric poset $[\vec{D}]$ on $\{1, 2, \ldots, n\}$, define its cyclic P-partition enumerator

$$F^{\text{cyc}}_{[\vec{D}]} := \sum_{f \in \mathcal{A}^{\text{tor}}([\vec{D}])} x_{f(1)} x_{f(2)} \cdots x_{f(n)}.$$

A special case yields the fundamental cyclic quasi-symmetric functions from Definition 2.3.

Proposition 3.13. *If $w \in \mathfrak{S}_n$ has $\text{cDes}(w) = J$, then $F^{\text{cyc}}_{[\vec{w}]} = F^{\text{cyc}}_{n,J}$.*

An immediate consequence of Lemma 3.11 is then the following.

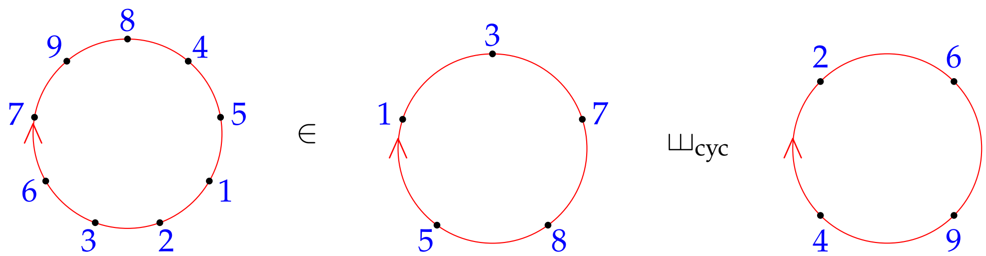


Figure 2: $[(8,4,5,1,2,3,6,7,9)] \in [(3,7,8,5,1)] \sqcup_{\mathrm{cyc}} [(6,9,4,2)]$.

Proposition 3.14. *For any toric poset $[\vec{D}]$, one has the following expansion*

$$F^{\mathrm{cyc}}_{[\vec{D}]} = \sum_{[\vec{w}] \in \mathcal{L}^{\mathrm{tor}}([\vec{D}])} F^{\mathrm{cyc}}_{n,\mathrm{cDes}(w)}.$$

We now use this fact to expand products of of basis elements $\{F^{\mathrm{cyc}}_{n,J}\}$ back in the same basis. The key notion is that of a cyclic shuffle of two total cyclic orders.

First recall the notion of a shuffle of permutations. For a finite set A of size a, let $\mathfrak{S}_A$ be the set of all bijections $w\colon [a] \to A$, viewed as words $w = (w_1, \ldots, w_a)$. Elements of $\mathfrak{S}_A$ will be called *bijective words*, a formal extension of permutations. Given two bijective words $u = (u_1, \ldots, u_a) \in \mathfrak{S}_A$ and $v = (v_1, \ldots, v_b) \in \mathfrak{S}_B$, where A and B are disjoint finite sets of integers, a bijective word $w \in \mathfrak{S}_{A \sqcup B}$ is a *shuffle* of u and v if u and v are subwords of w. Denote the set of all shuffles of u and v by $u \sqcup v$.

Definition 3.15. Let $C = A \sqcup B$ be a disjoint union of finite sets. Fix two total cyclic orders $[\vec{u}]$ and $[\vec{v}]$, with representatives $u = (u_1, \ldots, u_a) \in \mathfrak{S}_A$ and $v = (v_1, \ldots, v_b) \in \mathfrak{S}_B$. A total cyclic order $[\vec{w}]$, with $w \in \mathfrak{S}_C$, is a *cyclic shuffle of* $[\vec{u}]$ *and* $[\vec{v}]$ if there exists a representative $w' \in \mathfrak{S}_C$ of $[\vec{w}]$ which is (equivalently, every representative of $[\vec{w}]$ is) a shuffle of cyclic shifts of u and v, namely,

$$w' \in u' \sqcup v'$$

for some cyclic shift u' of u and cyclic shift v' of v.

Denote the set of all cyclic shuffles of $[\vec{u}]$ and $[\vec{v}]$ by $[\vec{u}] \sqcup_{\mathrm{cyc}} [\vec{v}]$.

Example 3.16. Let $A = \{1,3,5,7,8\}$ and $B = \{2,4,6,9\}$, and fix $u = (3,7,8,5,1) \in \mathfrak{S}_A$ and $v = (6,9,4,2) \in \mathfrak{S}_B$. An example of $[\vec{w}] \in [\vec{u}] \sqcup_{\mathrm{cyc}} [\vec{v}]$ is $[(8,4,5,1,2,3,6,7,9)]$, since $w' = (1,2,3,6,7,9,8,4,5)$ is a shuffle of $(1,3,7,8,5) \in [\vec{u}]$ and $(2,6,9,4) \in [\vec{v}]$. See Figure 2.

Observation 3.17. Let A and B be disjoint sets of integers, of cardinalities a and b respectively. For each $u = (u_1, u_2, \ldots, u_a) \in \mathfrak{S}_A$ and $v = (v_1, v_2, \ldots, v_b) \in \mathfrak{S}_B$ there are $\frac{(a+b-1)!}{(a-1)!(b-1)!}$ cyclic shuffles in $[\vec{u}] \sqcup\!\sqcup_{\text{cyc}} [\vec{v}]$.

We apply this setting to prove Theorem 1.2 and deduce the following consequences.

Proposition 3.18. cQSym *and* cQSym^- *are graded rings.*

Proposition 3.19. *The structure constants of* cQSym *and* cQSym^-*, with respect to the normalized fundamental basis, are nonnegative integers.*

4 Expansion of Schur functions in terms of fundamental cyclic quasi-symmetric functions

Theorem 1.3 follows from Theorem 4.4 below. The *cyclic descent map* on SYT of a given shape plays a key role in the proof; let us recall the relevant definition and main result from [2].

Definition 4.1 ([2, Definition 2.1])**.** Let $\mathcal{T}$ be a finite set, equipped with a *descent map* $\text{Des}\colon \mathcal{T} \longrightarrow 2^{[n-1]}$, where $n > 1$. A *cyclic extension* of Des is a pair (cDes, p), where $\text{cDes}\colon \mathcal{T} \longrightarrow 2^{[n]}$ is a map and $p\colon \mathcal{T} \longrightarrow \mathcal{T}$ is a bijection, satisfying the following axioms: for all T in $\mathcal{T}$:

$$\begin{aligned} \text{(extension)} \quad & \text{cDes}(T) \cap [n-1] = \text{Des}(T), \\ \text{(equivariance)} \quad & \text{cDes}(p(T)) = 1 + \text{cDes}(T), \\ \text{(non-Escher)} \quad & \varnothing \subsetneq \text{cDes}(T) \subsetneq [n]. \end{aligned}$$

Example 4.2. Let $\mathcal{T}$ be $\mathfrak{S}_n$, the symmetric group on n letters equipped with the classical descent map. The pair (cDes, p), with cDes defined as in (1.1) and p the cyclic shift, satisfies the axioms of Definition 4.1.

The notion of a descent set for a *standard Young tableau* T of skew shape λ/μ is well established (see, e.g., [11, p. 361]) . For the special case of *rectangular* shapes, Rhoades [10] constructed a cyclic extension satisfying the axioms of Definition 4.1. For almost all skew shapes there is a general existence result, as follows.

Theorem 4.3 ([2, Theorem 1.1])**.** *Let* λ/μ *be a skew shape with n cells. The descent map* Des *on* $\text{SYT}(\lambda/\mu)$ *has a cyclic extension* (cDes, p) *if and only if* λ/μ *is not a connected ribbon. Furthermore, for all* $J \subseteq [n]$*, all such cyclic extensions share the same cardinalities* $\#\text{cDes}^{-1}(J)$*.*

A constructive combinatorial proof of Theorem 4.3 was recently given in [8].

We shall now provide a cyclic analogue of the classical result [11, Theorem 7.19.7] (first proved in [6, Theorem 7]).

Theorem 4.4. *For every skew shape λ/μ of size n, which is not a connected ribbon, and for any cyclic extension* (cDes, p) *of* Des *on* $\mathrm{SYT}(\lambda/\mu)$*,*

$$s_{\lambda/\mu} = \sum_{A \in c2^{[n]}_{0,n}} m^{\mathrm{cyc}}(A)\, \widehat{F}^{\mathrm{cyc}}_{n,A}$$

where

$$m^{\mathrm{cyc}}(A) := m^{\mathrm{cyc}}(J) = \#\{T \in \mathrm{SYT}(\lambda/\mu) \,:\, \mathrm{cDes}(T) = J\} \qquad \left(\forall J \in A \in c2^{[n]}_{0,n}\right).$$

Recall *Postnikov's toric Schur functions* from [9].

Proposition 4.5. *For every non-hook shape λ, the coefficient of $\widehat{F}^{cyc}_{n,[J]}$ in s_λ is equal to the coefficient of s_λ in the Schur expansion of Postnikov's toric Schur function $s_{\mu(J)/1/\mu(J)}$.*

By [9, Theorem 5.3] these coefficients are equal to certain Gromov-Witten invariants.

5 Enumerative applications

Theorem 1.2 implies the following analogue of the shuffling theorem [12, Ex. 3.161] (see also [7, section 2.4]).

Proposition 5.1. *Let A and B be two disjoint sets of integers. For each $u \in \mathfrak{S}_A$ and $v \in \mathfrak{S}_B$, the distribution of the cyclic descent set over all cyclic shuffles of $[\vec{u}]$ and $[\vec{v}]$ depends only on* $\mathrm{cDes}([\vec{u}])$ *and* $\mathrm{cDes}([\vec{v}])$*.*

Consider now $\mathbb{Z}[[q]]$, the ring of formal power series in q, as a (free abelian) additive group with generators $(q^n)_{n=0}^{\infty}$, and define a new product by

$$q^i \odot q^j := q^{\max(i,j)},$$

extended linearly. We obtain a (commutative and associative) ring, to be denoted $\mathbb{Z}[[q]]_\odot$.

Consider also the ring $\mathbb{Z}[[\mathbf{x}]] = \mathbb{Z}[[x_1, x_2, \ldots]]$, and its subring $\mathbb{Z}[[\mathbf{x}]]_{\mathrm{bd}}$ consisting of bounded-degree power series. Define a map $\Psi : \mathbb{Z}[[\mathbf{x}]]_{\mathrm{bd}} \to \mathbb{Z}[[q]]_\odot$ by

$$\Psi(x_{i_1}^{m_1} \cdots x_{i_k}^{m_k}) := q^{i_k} \qquad (k > 0,\, i_1 < \cdots < i_k,\, m_1, \ldots, m_k > 0)$$

and $\Psi(1) := 1$, extended linearly.

Observation 5.2. Ψ is a ring ($\mathbb{Z}$-algebra) homomorphism.

Lemma 5.3. *For any positive integer n,*

$$\Psi(F^{\mathrm{cyc}}_{n,J}) = \frac{|J|q^{|J|} + (n - |J|)q^{|J|+1}}{(1-q)^n} = (1-q)\sum_r \binom{r+n-|J|-1}{n-1} r q^r \qquad (\forall J \subseteq [n]).$$

Using Theorem 1.2 and Lemma 5.3 we prove

Theorem 5.4. *Let A and B be two disjoint sets of integers with $|A| = m$ and $|B| = n$. For each $u \in \mathfrak{S}_A$ and $v \in \mathfrak{S}_B$, the distribution of the cyclic descent number over all cyclic shuffles of $[\overrightarrow{u}]$ and $[\overrightarrow{v}]$ is given by*

$$\sum_{[\overrightarrow{w}] \in [\overrightarrow{u}] \sqcup_{\mathrm{cyc}} [\overrightarrow{v}]} q^{\mathrm{cdes}(w)} = (1-q)^{m+n} \sum_r \binom{r+m-\mathrm{cdes}(u)-1}{m-1} \binom{r+n-\mathrm{cdes}(v)-1}{n-1} r q^r.$$

Theorem 5.4 implies Theorem 1.4. For other applications see the full version [1].

6 Open problems and final remarks

A Schur-positivity phenomenon, involving cyclic quasi-symmetric functions, was presented in Section 4. It is desired to find more results of this type. For example, it was proved in [5, Cor. 7.7] that, for any $0 < k < n$, the cyclic quasi-symmetric function

$$\sum_{\pi \in \mathfrak{S}_n \,:\, \mathrm{cdes}(\pi^{-1}) = k} F_{n,\mathrm{Des}(\pi)}$$

is symmetric and Schur-positive. Computational experiments suggest the following refined cyclic version.

Conjecture 6.1. *For every $\varnothing \subsetneq J \subsetneq [n]$ the cyclic quasi-symmetric function*

$$\sum_{\substack{\pi \in \mathfrak{S}_n \\ [\mathrm{cDes}(\pi^{-1})] = [J]}} F^{\mathrm{cyc}}_{n,\mathrm{cDes}(\pi)} = \sum_{\substack{\pi \in \mathfrak{S}_n \\ (\exists i)\ \mathrm{cDes}(\pi^{-1}) = J+i}} F^{\mathrm{cyc}}_{n,\mathrm{cDes}(\pi)}$$

is symmetric and Schur-positive.

Cyclic descents were introduced by Cellini [3] in the search for subalgebras of Solomon's descent algebra. An important subalgebra of the descent algebra is the peak algebra.

Problem 6.2. *Define and study cyclic peaks and a cyclic peak algebra.*

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