Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #59, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

A new formula for Stanley's chromatic symmetric function for unit interval graphs and e-positivity for triangular ladder graphs

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Abstract. In 1995 Stanley conjectured that the chromatic symmetric functions of the graphs $P_{d,2}$, which we call triangular ladders, are e-positive. In this extended abstract we summarize our confirmation of this conjecture, which is also an unsolved case of the celebrated $(3+1)$-free conjecture. Gebhard and Sagan defined chromatic symmetric functions in non-commuting variables that satisfy a deletion-contraction property unlike the chromatic symmetric functions in commuting variables. We prove a new signed combinatorial formula for the chromatic symmetric function of *any* unit interval graph in the basis of elementary symmetric functions. Then we find that triangular ladders are e-positive by very carefully defining a sign-reversing involution on our signed combinatorial formula, which leaves us with certain positive terms and further allows us to expand an already known family of e-positive graphs by Gebhard and Sagan.

Keywords: symmetric functions, graph colorings, unit interval graphs

1 Introduction

The chromatic symmetric function X_G of a simple graph G, defined by Richard Stanley [17], is a generalization of the chromatic polynomial defined by Birkoff [2] and has received a lot of attention of late. These symmetric functions retain many properties of chromatic polynomials, including evaluating to the number of acyclic orientations [17], but do not satisfy a useful deletion-contraction property like chromatic polynomials do. However, they do have connections to representation theory and algebraic geometry [12], which has been a further motivation in their study and particularly behind the study of their e-positivity and Schur-positivity. A symmetric function is *e-positive* (respectfully *Schur-positive*) if X_G can be written as a positive linear combination of elementary (respectfully Schur) symmetric functions. Throughout this abstract we refer to the graph as e-positive (respectfully Schur-positive) if X_G is so.

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In 1995 Stanley [17] conjectured that if a poset is $(3+1)$-free then its incomparability graph is e-positive, which is equivalent to the Stanley-Stembridge conjecture in 1993 [18]. Guay-Paquet [10] reduced this conjecture to showing that the incomparability graphs of $(3+1)$ and $(2+2)$-free posets are e-positive. These types of graphs are known as unit interval graphs and have a connection to Jacobi-Trudi matrices [18]. Gasharov [8] has proven that the incomparability graph of a $(3+1)$-free poset is Schur-positive, which is weaker than the full conjecture since e-positivity implies Schur-positivity.

There have been some generalizations and further partial results on the $(3+1)$-free conjecture. In 1995 Stanley proved that paths and cycles are e-positive [17] and their functions are explicitly described in [20]. Coefficients of other graphs have been studied in [13, 14]. Other works have focused on finding graph properties relating to e-positivity with an emphasis on induced subgraphs [5, 11, 19]. Shareshian and Wachs [16] defined a generalization of the chromatic symmetric function in the algebra of quasi-symmetric functions. These also do not satisfy a deletion-contraction property, but do generalize the $(3+1)$-free conjecture and are conjectured to be e-unimodal. This has further been independently generalized by Ellzey [6] and Alexandersson and Panova [1] to circular indifference graphs. Cho and Huh [3] provide a new family of e-positive graphs along with a new proof to an old family that was first proven to be e-positive by Stanley [17].

In this abstract we summarize our results about the graphs, $P_{d,2}$, which are specifically mentioned in Stanley's original 1995 paper [p190, 16] where he wrote

"It remains open whether $P_{d,2}$ is e-positive".

In order to do this, we use a generalization of X_G by Gebhard and Sagan [9] to symmetric functions in non-commuting variables, Y_G, that satisfies a deletion-contraction property. Gebhard and Sagan find a new family of e-positive graphs by semi-symmetrizing their chromatic symmetric functions in non-commuting variables. We use ideas from their paper and expand their proven family of e-positive graphs. This expansion will include all $P_{d,2}$, which we call *triangular ladders*. First we prove a new signed combinatorial formula for *all* unit interval graphs in the basis of elementary symmetric functions. Then we prove the e-positivity for triangular ladders by very carefully defining a sign-reversing involution on the associated signed set that leaves us with certain positive terms.

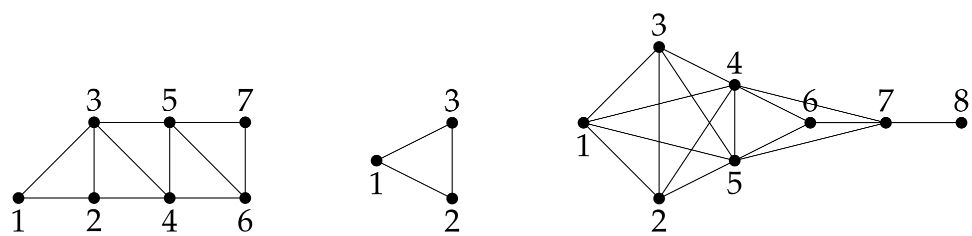
In Section 2 we describe some of the background we will need including the definition of unit interval graphs and Gebhard and Sagan's deletion-contraction property in non-commuting variables. In Section 3 we describe our signed combinatorial formula for expanding the chromatic symmetric function of any unit interval graph in the elementary basis. Our method is to repeatedly use the deletion-contraction property until we have no edges and then reinterpret the coefficients in a combinatorial manner using arc diagrams with arc markings, vertex labels and vertex markings. In Section 4 we apply our signed combinatorial formula to triangular ladders and summarize our results about a new family of e-positive graphs that expands a family proven to be e-positive by

Gebhard and Sagan [9]. Lastly, at the end of Section 4 we describe the fixed points for the sign-reversing involution. Full results are in [4].

2 Background

There are many equivalent definitions for unit interval graphs with some equivalences proven in [7]. Here we will describe *unit interval graphs* on vertices in $[n] = \{1, 2, \ldots, n\}$ using a collection of intervals $[a_1, b_1]$, $[a_2, b_2]$, $\ldots$, $[a_l, b_l]$ where $a_k \leq b_k$ are in $[n]$ with $[a, b] = \{a, a+1, \ldots, b\}$. Note that intervals can be chosen inefficiently. The graph has an edge from i to j whenever $i, j \in [a_k, b_k]$ for some k. In the literature there are special families of unit interval graphs $P_{n,k}$ that are formed from the intervals $[1, k+1]$, $[2, k+2]$, $\ldots$, $[n-k, n]$ and this is the notation Stanley uses in his paper [17]. Many well-known families of graphs including the *complete graphs*, $K_n = P_{n,n-1}$, and the *paths*, $P_n = P_{n,1}$ are unit interval graphs. Our focus is on when $k = 2$, $P_{n,2}$, which we call the *triangular ladders*, TL_n.

Example 1. From left to right we have $TL_7 = P_{7,2}$, $K_3 = P_{3,2}$ and the unit interval graph formed from the intervals $[1, 5]$, $[4, 7]$ and $[7, 8]$.



The *algebra of symmetric functions in non-commuting variables*, NCSym, is generated by several classical bases, all of which are indexed by set partitions. A *set partition*, $\pi = B_1/B_2/\cdots/B_l$ of $[n]$, denoted $\pi \vdash [n]$, is a collection of non-empty disjoint subsets $B_i \subseteq [n]$ called *blocks* that union to form the full set $[n]$. Rosas and Sagan [15] define all the classical functions and give transition formulas between them. Our interest is in the elementary basis. Given $\pi \vdash [n]$ the *elementary symmetric function in non-commuting variables*, e_π, is

$$e_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n},$$

which is summed over tuples $(i_1, i_2, \ldots, i_n)$ of positive integers where $i_j \neq i_k$ if j and k are in the same block of π.

Example 2. For $12 \vdash [2]$ and $12/3 \vdash [3]$ we have

$$e_{12} = x_1x_2 + x_2x_1 + x_1x_3 + x_3x_1 + x_2x_3 + x_3x_2 + \cdots \text{ and}$$

$$e_{12/3} = x_1x_2x_1 + x_2x_1x_2 + \cdots + x_1x_2x_2 + x_2x_1x_1 + \cdots + x_1x_2x_3 + x_2x_1x_3 + \cdots.$$

Though we work with NCSym, we are interested in symmetric functions. Define $\rho : \mathrm{NCSym} \to \Lambda$ to be the *commuting map* where $f \in \mathrm{NCSym}$ is mapped to f but we let the variables commute.

Example 3. We have $\rho(e_{12}) = 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \cdots$.

The bases for the *algebra of symmetric functions*, Λ, are indexed by *integer partitions*, $\lambda = \lambda_1\lambda_2 \dots \lambda_l$, weakly decreasing lists of positive integers. We write $\lambda \vdash n$ if all the λ_i sum to n. The *ith elementary symmetric function in commuting variables* is

$$e_i = \sum_{j_1 < j_2 < \cdots < j_i} x_{j_1} x_{j_2} \cdots x_{j_i}$$

where for an integer partition $\lambda = \lambda_1\lambda_2 \dots \lambda_l$ we define the *elementary symmetric function*, e_λ, to be

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}.$$

Example 4. For $2 \vdash 2$ and $21 \vdash 3$ we have

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 + \cdots \text{ and } e_{21} = e_2e_1 = x_1^2x_2 + x_1x_2^2 + \cdots + 3x_1x_2x_3 + \cdots .$$

The elementary symmetric functions in NCSym and Λ are closely related. Define for a set partition $\pi \vdash [n]$ the integer partition $\lambda(\pi) \vdash n$, which is formed from the sizes of all the blocks in π. For example, $\lambda(14/235/67) = 322$. Rosas and Sagan [15] proved for $\pi \vdash [n]$ that $\rho(e_\pi) = \pi! e_{\lambda(\pi)}$ where $\pi! = \lambda(\pi)! = \lambda_1!\lambda_2! \cdots \lambda_l!$. We will call a function $f \in \Lambda$ *e-positive* if f can be written as a non-negative sum of elementary symmetric functions.

A *proper coloring* κ of a graph G with vertex set V is a function

$$\kappa : V \to \{1, 2, \dots\}$$

such that if $u, v \in V$ are adjacent, then $\kappa(u) \neq \kappa(v)$. Fixing an order for the vertices the *chromatic symmetric function in non-commuting* is defined to be

$$Y_G = \sum_\kappa x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

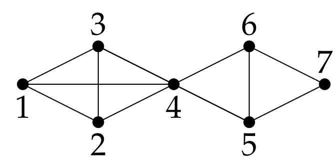
where the sum is over all proper colorings κ of G. Because the variables don't commute the chosen order for vertices matters. If we let the variables commute then we get the *chromatic symmetric function in commuting variables*, which we denote $X_G = \rho(Y_G)$. We will call a graph G itself *e-positive* if X_G is e-positive.

Example 5. All complete graphs K_n are e-positive with

$$Y_{G_n} = e_{12\cdots n} \text{ and } \rho(Y_{K_n}) = X_{K_n} = n!e_n.$$

The main result we present in this abstract is about a new family of e-positive graphs. This family of graphs is formed by combining complete graphs and triangular ladders in a certain way. Given a graph G with labels in $[n]$ and a graph H with labels in $[m]$ we define their *concatenation* to be the graph $G \cdot H$ on vertices $[n+m-1]$ where the graph on the first n vertices is isomorphic to G, the graph on the last m vertices is isomorphic to H and no additional edges are included.

Example 6. For example $K_4 \cdot TL_4$ is below.



Though the X_G don't satisfy a deletion-contraction property Gebhard and Sagan [9] showed that Y_G does. We define the *deletion* of an edge ϵ of G, $G \setminus \epsilon$, to be the graph G with edge ϵ removed. The *contraction* of an edge ϵ between vertices i and j is the graph G after merging the vertices i and j and any multiedges created into a single edge.

Example 7. For $G = P_3$ and ϵ the edge between 2 and 3 we have $G \setminus \epsilon$ to be the unit interval graph on [3] with interval $[1,2]$ and $G/\epsilon = P_2$.

Roughly speaking, X_G fails to have a deletion-contraction property due to its homogeneous degree where $X_{G\setminus\epsilon}$ and $X_{G/\epsilon}$ have different degrees. In non-commuting variables we can compensate for this. Define the *induced* monomial for $j < n$ to be

$$x_{i_1}x_{i_2}\ldots x_{i_j}\ldots x_{i_{n-1}} \uparrow_j = x_{i_1}x_{i_2}\ldots x_{i_j}\ldots x_{i_{n-1}}x_{i_j}$$

where a copy of the jth variable is included at the end and extend this definition linearly.

Example 8. We have

$$\begin{aligned} e_{12}\uparrow_2 &= x_1x_2x_2 + x_2x_1x_1 + x_1x_3x_3 + x_3x_1x_1 + x_2x_3x_3 + x_3x_2x_2 + \cdots \\ &= \frac{1}{2}(e_{12/3} + e_{13/2} - e_{1/23} - e_{123}). \end{aligned}$$

Proposition 9 (Deletion-Contraction, Gebhard and Sagan [9] Proposition 3.5)**.** For G with vertices $V = [n]$ and an edge ϵ between vertices j and n we have

$$Y_G = Y_{G\setminus\epsilon} - Y_{G/\epsilon}\uparrow_j .$$

Example 10. We can use deletion-contraction on P_3 on edge ϵ between 2 and 3 to get

$$Y_{P_3} = Y_{K_2}Y_{K_1} - Y_{K_2}\uparrow_2 = e_{12/3} - e_{12}\uparrow_2 .$$

The formula for an induced e_π generally has many terms, but after symmetrizing many of these terms cancel out. Gebhard and Sagan define equivalence classes on set partitions that enable us to partially symmetrize functions. We will say two set partitions $\pi \vdash [n]$ and $\sigma \vdash [n]$ are related, $\pi \sim \sigma$, if

1. $\lambda(\pi) = \lambda(\sigma)$ and
2. if A and B are blocks of π and σ respectively where $n \in A$ and $n \in B$ then $|A| = |B|$.

Let

$$(\pi) = \{\sigma : \sigma \sim \pi\}. \tag{2.1}$$

Example 11. The only non-singleton equivalence class for the set partitions of [3] is $(13/2) = \{13/2, 1/23\}$ and also the equivalence class for $1/234$ is

$$(1/234) = \{1/234, 134/2, 124/3\}.$$

Two functions $f, g \in \text{NCSym}$ are *equivalent*, $f \equiv g$, if the sum of the coefficients of the terms associated to π in the same equivalence class are equal in the elementary basis.

Example 12. Because $13/2 \sim 1/23$

$$Y_{P_3} = \frac{1}{2}(e_{12/3} - e_{13/2} + e_{1/23} + e_{123}) \equiv \frac{1}{2}(e_{12/3} + e_{123}).$$

Our study in [4], summarized here, is about whether graphs G themselves are e-positive, which is a question about X_G in fully commuting variables. Just like in Gebhard and Sagan's paper we show that Y_G is e-positive after partially symmetrizing variables along the lines of these equivalence classes. To formalize this, we call $f \in \text{NCSym}$ *semi-symmetrized e-positive* if $f \equiv g$ for some $g \in \text{NCSym}$ that can be written as a non-negative sum of elementary symmetric functions in non-commuting variables. We call a graph G *semi-symmetrized e-positive* if Y_G is semi-symmetrized e-positive. It follows that if Y_G is semi-symmetrized e-positive then certainly $\rho(Y_G) = X_G$ is e-positive and G is e-positive. Semi-symmetrized e-positivity is a stronger condition than e-positivity.

Using partial symmetrizing, Gebhard and Sagan have a nice formula for inducing e_π. Given an integer partition $\pi \vdash [n-1]$ for $j < n$ define $\pi \oplus_j n \vdash [n]$ to be the integer partition where we place n in the same block as j. For example $14/23 \oplus_4 5 = 145/23$.

Proposition 13 (Gebhard and Sagan [9] Corollary 6.1)**.** For $\pi \vdash [n-1]$, $j < n$ and b the size of the block in π containing $n-1$ we have

$$e_\pi \uparrow_j \equiv \frac{1}{b}(e_{\pi/n} - e_{\pi \oplus_j n}).$$

Example 14. To continue our deletion-contraction example for P_3 we have

$$Y_{P_3} = e_{12/3} - e_{12} \uparrow_2 \equiv e_{12/3} - \frac{1}{2}(e_{12/3} - e_{123}) = \frac{1}{2}(e_{12/3} + e_{123}).$$

3 A new formula for unit interval graphs

Let us delete and contract a unit interval graph G until we have a sum of graphs with no edges. If Y_G is calculated similarly in the power-sum basis we will arrive at an example of Stanley's broken-circuit theorem ([17] Theorem 2.9). Since our interest is in the elementary basis we will use Gebhard and Sagan's [9] formula in Proposition 13, which will give us a new signed combinatorial formula. In the case of triangular ladders, we can define a sign-reversing involution and simplify the sum to only positive terms.

Theorem 15. *Given a unit interval graph on n vertices with intervals $[a_1, 1], [a_2, 2], \ldots, [a_n, n]$ let G' be the same graph on $[n-1]$ after removing vertex n. Then,*

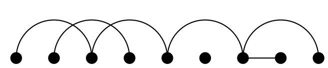
$$Y_G = Y_{G'} Y_{K_1} - \sum_{i=a_n}^{n-1} Y_{G'} \uparrow_i .$$

If we repeatedly use the formula in Theorem 15 we will have a sum of induced Y_H where H is a graph with no edges. We will associate each induced H to an arc diagram where each induction will be represented as an arc. An *arc diagram* is a drawing on n vertices in a line numbered from left to right together with a collection of arcs (i, j), $i < j$. The *length of a diagram D*, $\ell(D)$, is the number of vertices minus one. Define an arc (i, j) to be a *left arc* of j. The collection of arc diagrams, $\mathcal{A}(G)$, for a unit interval graph G with intervals $[a_1, 1], [a_2, 2], \ldots, [a_n, n]$ are those where

- all vertices have at most one left arc and
- if we have an arc (i, j) then $i, j \in [a_k, k]$ for some k.

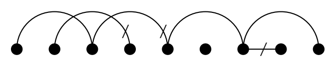
Given any arc diagram $D \in \mathcal{A}(G)$ define $a(D)$ to be the number of arcs in the arc diagram D and $\pi(D)$ to be the set partition formed by the connected components of D.

Example 16. The diagram $D \in \mathcal{A}(TL_9)$ below has $\pi(D) = 135789/24/6$ and $a(D) = 6$.



Recall that when we induce an elementary symmetric function once by Proposition 13 we have a difference of two elementary symmetric functions after semi-symmetrizing. Each single induction of Y_H is associated to a single arc in its arc diagram, and we will keep track of the two subtracted terms by marking the arcs with tic marks. A tic mark on arc (i, j) will split its block into *pieces*, refining the initial set partition $\pi(D)$. Every dot to the right of j including j will be in a different piece then those to the left of j. For an arc diagram D' with possible tic marks on arcs define $\pi(D')$ to be the set partition whose blocks are the pieces and $t(D')$ to be the number of tic marks on the diagram D'.

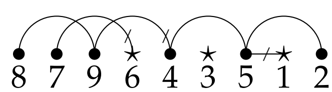
Example 17. The diagram D' below has $a(D') = 6$, $t(D') = 3$ and $\pi(D') = 13/2/4/57/6/89$.



After inducing all the Y_H and using the formula in Proposition 13 we have a function in NCSym equivalent to Y_G. We can combinatorially reinterpret the coefficients of this function as the following set $\mathcal{A}'_L(G)$ of arc diagrams. All $D' \in \mathcal{A}'_L(G)$ have

- an underlying arc diagram in $\mathcal{A}(G)$,
- each arc has a possible tic mark that breaks the connected component into pieces,
- a permutation labeling $\delta \in \mathfrak{S}_n$ where $\mathfrak{S}_n$ is the set of permutations on $[n]$,
- the permutation labeling increases from left to right on all pieces and
- one of the vertices in the right-most piece of each connected component is marked with a star.

Example 18. Below we have $D' \in \mathcal{A}'_L(TL_9)$ with permutation label $879643512 \in \mathfrak{S}_9$ and $a(D') = 6$, $t(D') = 3$ and $\pi(D') = 13/2/4/57/6/89$.



Example 19. The set $\mathcal{A}'_L(TL_2)$ has 6 elements listed below.

Theorem 20. *For a unit interval graph G on n vertices,*

$$Y_G \equiv \frac{1}{n!} \sum_{D' \in \mathcal{A}'_L(G)} (-1)^{t(D')} e_{\pi(D')}.$$

Example 21. Using Theorem 20 on $G = TL_2$ we have

$$Y_{TL_2} \equiv \frac{1}{2!} \left(e_{12} + e_{12} + e_{1/2} + e_{1/2} - e_{1/2} - e_{1/2} \right) = e_{12}.$$

4 Triangular ladders

Defining a sign-reversing involution on the signed-combinatorial set $\mathcal{A}'_L(TL_n)$ will prove that TL_n is e-positive. This involution will allow us to further conclude that all concatenations of complete graphs and triangular ladders are e-positive. Full details are in [4].

Theorem 22. *The triangular ladder TL_n, $n \geq 1$, is semi-symmetrized e-positive and so e-positive.*

The involution that proves Theorem 22 can be used to prove that if G is a semi-symmetrized e-positive graph then so is $G \cdot TL_n$, which extends the following result.

Theorem 23 (Gebhard and Sagan [9] Theorem 7.6 and 7.8)**.** *If a graph G is semi-symmetrized e-positive then so is the concatenation $G \cdot K_m$ and $G \cdot TL_4$.*

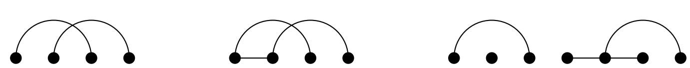
Proposition 24. Any graph G such that

$$G = G_1 \cdot G_2 \cdots G_l,$$

where $G_i = TL_{n_i}$ or $G_i = K_{n_i}$, is a semi-symmetrized e-positive graph, so also e-positive.

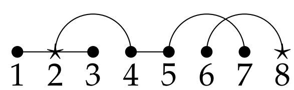
We end this abstract by describing the fixed points of our sign-reversing involution $\varphi : \mathcal{A}'_L(TL_n) \to \mathcal{A}'_L(TL_n)$, which requires some technical definitions. The *concatenation* of two arc diagrams D_1 on vertices $[n]$ and D_2 on vertices $[m]$ is $D_1 \cdot D_2$, the arc diagram on $[n+m-1]$ where D_1 is on the first n vertices and D_2 is on the last m. For a diagram D denote $D \cdot D \cdots D$ by D^m where we have m copies of D. Arc diagrams associated to the triangular ladders are concatenations of two kinds of arc diagrams. One is an *interlacing arc diagram*, IL, on $n \geq 2$ vertices, which has arcs $(i, i+2)$ for all possible i. Let L_m be the length m IL diagram. The other is an *interconnecting arc diagram*, IC, which is an IL diagram, but with the arc $(1,2)$ included. Let C_m be the length m IC diagram.

Example 25. From left to write we have L_3, C_3 and $L_2 \cdot L_1 \cdot C_1 \cdot C_2$.



Note that IL diagrams of length 1 naturally break our diagrams into *sections*, which are diagrams without any L_1 in its decomposition. A section satisfies *the IC-condition* if it contains a C_m, $m \geq 2$, in its decomposition with only C_1 to its left. Consider a $D \in \mathcal{A}'_L(TL_n)$ with no tic marks that ends in $P = L_{2k-1}$, $k \geq 1$. On vertex $n-1$ we have a right endpoint of a connected component, so one vertex in this component will be marked with a star. Write $e(D) = i$ if the ith right-most vertex in the component is marked with a star. From the $(n-1)$st vertex we can count the number of vertices going left in the same connected component until we reach a C_m or L_m with $m \geq 2$. If we counted i vertices including the right-most vertex of C_m or L_m then let $s(D) = i + 1$. If instead we counted until the left endpoint let $s(D)$ be the number of vertices in the connected component of $n - 1$. For ease we will write $e(P)$ for $e(D)$ and $s(P)$ for $s(D)$, assuming that we are looking at P in the larger scope of diagram D.

Example 26. Diagram $D \in \mathcal{A}'_L$ below satisfies the IC-condition, has only one section, $s(D) = 4$, $e(D) = 5$ and $\pi(D) = 123457/68$.



Proposition 27. The fixed points are diagrams $D \in \mathcal{A}'_L$ with no tic marks with the following conditions. The diagram above satisfies all five conditions.

FP1. All sections satisfy the IC-condition except for the right-most ending at n if it is C_1^k, $k \geq 0$.

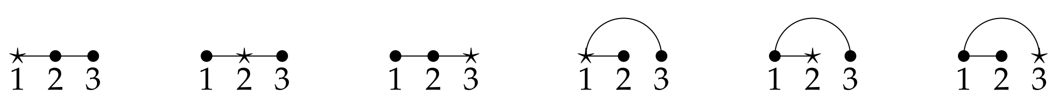
FP2. All $P = L_{2k-1}$, $k \geq 1$, in the decomposition have $e(P) \notin [a(P)/2 + 1, s(P) - 2]$.

FP3. All $P = L_{2k-1}$, $k \geq 1$, in the decomposition with $e(P) = s(P) - 1$ are immediately preceded by $L_{2j} \cdot C_1^m$, $j, m \geq 1$, with $e(L_{2j}) = s(L_{2j})$.

FP4. All $P = L_{2k-1}$, $k \geq 1$, in the decomposition that end before n have $e(P) \neq s(P)$.

FP5. All $P = L_k$, $k \geq 1$, in the decomposition that are followed by $P = L_m$, $m \geq 1$, have $e(P) \neq s(P)$.

Example 28. There are 6 fixed points for TL_3 listed below.



By computer computation all unit interval graphs up to 7 vertices are semi-symmetrized e-positive. It can be conjectured that all unit interval graphs satisfy this stronger condition. Also, by Stanley's work the fixed points mentioned above should have a connection to acyclic orientations. For these two reasons there may be more to learn in this direction.

Acknowledgements

The author would like to thank Stephanie van Willigenburg for bringing this open problem to her attention and for valuable comments and helpful conversations. She would also like to thank the reviewers and Susanna Fishel for their advice and comments.

References

[1] P. Alexandersson and G. Panova. "LLT polynomials, chromatic quasisymmetric functions and graphs with cycles". *Discrete Math.* **341**.12 (2018), pp. 3453–3482. Link.

[2] G. Birkhoff. "A determinant formula for the number of ways of coloring a map". *Ann. of Math. (2)* **14**.1-4 (1912), pp. 42–46. Link.

[3] S. Cho and J. Huh. "On e-positivity and e-unimodality of chromatic quasisymmetric functions". 2017. arXiv:1711.07152.

[4] S. Dahlberg. "Triangular ladders $P_{d,2}$ are e-positive". 2018. arXiv:1811.04885.

[5] S. Dahlberg, A. Foley, and S. van Willigenburg. "Resolving Stanley's e-positivity of claw-contractible-free graphs". 2017. arXiv:1703.05770.

[6] B. Ellzey. "Chromatic quasisymmetric functions of directed graphs". *Sém. Lothar. Combin.* **78B** (2017), pp. 1–12. Link.

[7] B. Ellzey. *On the Chromatic Quasisymmetric Functions of Directed Graphs*. Thesis (Ph.D.)–University of Miami. ProQuest LLC, Ann Arbor, MI, 2018, p. 146. Link.

[8] V. Gasharov. "On Stanley's chromatic symmetric function and clawfree graphs". *Discrete Math.* **205**.1-3 (1999), pp. 229–234. Link.

[9] D. Gebhard and B. Sagan. "A chromatic symmetric function in noncommuting variables". *J. Algebraic Combin.* **13**.3 (2001), pp. 227–255. Link.

[10] M. Guay-Paquet. "A modular relation for the chromatic symmetric functions of $(3+1)$-free posets". 2013. arXiv:1306.2400.

[11] A. Hamel, C. Hoàng, and J. Tuero. "Chromatic symmetric functions and H-free graphs". 2017. arXiv:1709.03354.

[12] M. Harada and M. Precup. "The cohomology of abelian Hessenberg varieties and the Stanley-Stembridge conjecture". 2017. arXiv:1709.06736.

[13] A. Paunov. "Planar graphs and Stanley's chromatic functions". 2017. arXiv:1702.05787.

[14] A. Paunov and A. Szenes. "A new approach to e-positivity for Stanley's chromatic functions". 2017. arXiv:1702.05791.

[15] M. Rosas and B. Sagan. "Symmetric functions in noncommuting variables". *Trans. Amer. Math. Soc.* **358**.1 (2006), pp. 215–232. Link.

[16] J. Shareshian and M. Wachs. "Chromatic quasisymmetric functions". *Adv. Math.* **295** (2016), pp. 497–551. Link.

[17] R. Stanley. "A symmetric function generalization of the chromatic polynomial of a graph". *Adv. Math.* **111**.1 (1995), pp. 166–194. Link.

[18] R. Stanley and J. Stembridge. "On immanants of Jacobi-Trudi matrices and permutations with restricted position". *J. Combin. Theory Ser. A* **62**.2 (1993), pp. 261–279. Link.

[19] S. Tsujie. “The chromatic symmetric functions of trivially perfect graphs and cographs”. *Graphs Combin.* **34**.5 (2018), pp. 1037–1048. Link.

[20] M. Wolfe. “Symmetric chromatic functions”. *Pi Mu Epsilon* **10**.8 (1998), pp. 643–657. Link.