Candidate for the crystal $B(-\infty)$ for the queer Lie superalgebra

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Abstract

It is shown that the direct limit of the semistandard decomposition tableau model for polynomial representations of the queer Lie superalgebra exists, which is believed to be the crystal for the upper half of the corresponding quantum group. An extension of this model to describe the direct limit combinatorially is given. Furthermore, it is shown that the polynomials representations may be recovered from the limit in most cases.

Quantum groups and Cartan data

Let $I_0 = \{1, \dots, n-1\}$ and $I = I_0 \sqcup \{\overline{1}\}$. Denote the standard basis vectors of $\mathbf{Z}^n$ by $\epsilon_1, \dots, \epsilon_n$ and define $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for each $i \in I_0$. Set

$$\Lambda^- = \left\{ \lambda = -\sum_{i=1}^n \lambda_i \epsilon_i \in \mathbf{Z}^n_{\le 0} : \begin{array}{l} \lambda_i \ge \lambda_{i+1} \text{ and } \lambda_i = \lambda_{i+1} \text{ implies} \\ \lambda_i = \lambda_{i+1} = 0 \text{ for all } i = 1, \dots, n \end{array} \right\}.$$

Equip Λ^- with a partial order $\lambda \le \mu$ if and only if $\mu - \lambda \in \Lambda^-$. An element $\lambda = -\lambda_1\epsilon_1 - \cdots - \lambda_n\epsilon_n$ in Λ^- will be henceforth be identified with the strict partition $w_0\lambda = (\lambda_n, \dots, \lambda_1)$.

Abstract $\mathfrak{q}(n)$-crystals

Definition (Grantcharov *et al.*, 2015)

An *abstract $\mathfrak{q}(n)$-crystal* is an abstract $\mathfrak{gl}(n)$-crystal $\mathcal{B}$ together with maps $e_{\overline{1}}, f_{\overline{1}} \colon \mathcal{B} \longrightarrow \mathcal{B} \sqcup \{0\}$ such that

1. $\operatorname{wt}(\mathcal{B}) \subseteq \mathbf{Z}^n_{\ge 0}$;
2. $\operatorname{wt}(e_{\overline{1}}b) = \operatorname{wt}(b) + \alpha_1$ provided $e_{\overline{1}}b \ne 0$;
3. $\operatorname{wt}(f_{\overline{1}}b) = \operatorname{wt}(b) - \alpha_1$ provided $f_{\overline{1}}b \ne 0$;
4. for any $b, b' \in \mathcal{B}$, $f_{\overline{1}}b = b'$ if and only if $b = e_{\overline{1}}b'$;
5. if $3 \le i \le n-1$, we have
   (a) the operators $e_{\overline{1}}$ and $f_{\overline{1}}$ commute with e_i and f_i,
   (b) if $e_{\overline{1}}b \in \mathcal{B}$, then $\varepsilon_i(e_{\overline{1}}b) = \varepsilon_i(b)$ and $\varphi_i(e_{\overline{1}}b) = \varphi_i(b)$.

Decomposition tableaux

Definition (Grantcharov *et al.*, 2014)

Let $\lambda = (\lambda_n, \dots, \lambda_1)$ be a strict partition. Define $\ell(\lambda)$ to be the number of $1 \le i \le n$ such that $\lambda_i \ne 0$.

1. The *shifted Young diagram of shape λ* is an array of boxes in which the i-th row has λ_{n+1-i} cells, and is shifted $i-1$ units to the right with respect to the top row.
2. A word $u = u_1u_2\cdots u_N$ is a *hook word* if there exists $1 \le k \le N$ such that
$$u_1 \ge u_2 \ge \cdots \ge u_k < u_{k+1} < \cdots < u_N.$$
3. A *semistandard decomposition tableau of shifted shape λ* is a filling T of λ with letters from $\{1, 2, \dots, n\}$ such that
   (a) the word v_i formed by reading the i-th row from left to right is a hook word of length λ_{n-i+1}, and
   (b) v_i is a hook subword of maximal length in $v_{i+1}v_i$ for $1 \le i \le \ell(\lambda) - 1$.
4. Set $\operatorname{read}(T)$ to be the word obtained by reading T in rows from right to left starting at the top.
5. For $\lambda \in \Lambda^-$, let $\mathrm{SDT}(\lambda)$ denote the set of all semistandard decomposition tableaux of shape $w_0\lambda$.

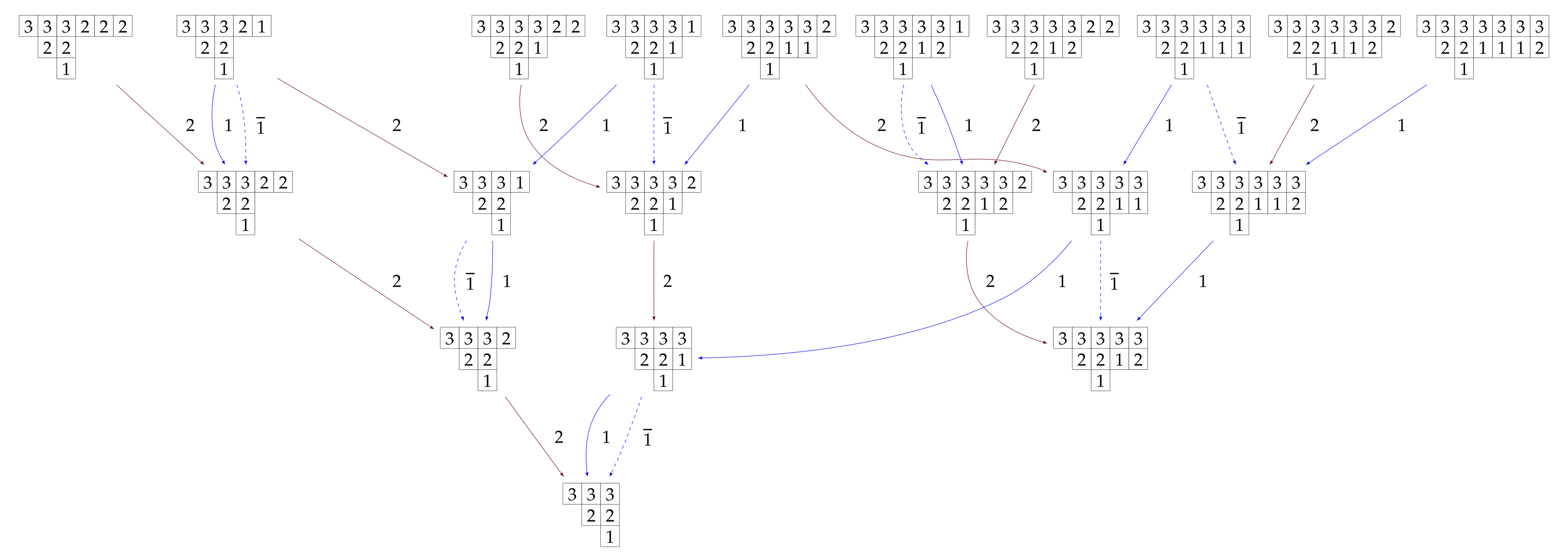
References

[1] M. Gillespie, G. Hawkes, W. Poh, and A. Schilling, *Characterization of queer supercrystals*, 2018.

[2] D. Grantcharov, J. H. Jung, S.-J. Kang, M. Kashiwara, and M. Kim, *Crystal bases for the quantum queer superalgebra and semistandard decomposition tableaux*, Trans. Amer. Math. Soc. **366** (2014), no. 1, 457–489.

[3] ______, *Crystal bases for the quantum queer superalgebra*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 7, 1593–1627.

Bottom of $B(-\infty)$ for type $\mathfrak{q}(3)$



$\mathfrak{q}(n)$-crystals

Definition (Grantcharov *et al.*, 2014; Gillespie *et al.*, 2018)

Let T be a semistandard decomposition tableau of shape $w_0\lambda$.

1. Suppose $i \in I_0$. Then e_iT and f_iT are computed using $\mathfrak{gl}(n)$-crystal rules.
2. If $i = -1$, consider the subword w of $\operatorname{read}(T)$ consisting of only the letters 1 and 2.
   (a) If the leftmost letter in w is 1, then $e_{\overline{1}}T = 0$. Otherwise $e_{\overline{1}}T$ is the tableau obtained from T by changing the 2-box corresponding to the leftmost 2 in w to a 1-box.
   (b) If the leftmost letter in w is 2, then $f_{\overline{1}}T = 0$. Otherwise $f_{\overline{1}}T$ is the tableau obtained from T by changing the 1-box corresponding to the leftmost 1 in w to a 2-box.

For a $\lambda \in \Lambda^-$, define $L^\lambda \in \mathrm{SDT}(\lambda)$ to be the tableau whose i-th row from the bottom contains only the letter i.

Theorem (Grantcharov *et al.*, 2014)

For $\lambda \in \Lambda^-$, the set $\mathrm{SDT}(\lambda)$ together with the operators defined above form an abstract $\mathfrak{q}(n)$-crystal isomorphic to the crystal of the irreducible highest weight $\mathfrak{q}(n)$-module with highest weight $w_0\lambda$. Moreover, a unique lowest weight vector in $\mathrm{SDT}(\lambda)$ is L^λ.

Candidate for $B(-\infty)$

Denote the set of all *dual marginally large* semistandard tableaux for $\mathfrak{q}(n)$ by $\mathrm{SDT}(-\infty)$. For $i \in I$, e_i and f_i may be defined as on $\mathrm{SDT}(\lambda)$ with the additional requirement that dual marginal largeness must be preserved by pushing in/out "trivial" columns.

Theorem (Salisbury–Scrimshaw)

1. Suppose $\lambda \le \mu$. Then there exists a $\mathfrak{q}(n)$-crystal embedding
$$\mathrm{SDT}(\lambda) \otimes \mathcal{T}_{-\lambda} \hookrightarrow \mathrm{SDT}(\lambda + \mu) \otimes \mathcal{T}_{-\lambda-\mu}$$
such that $L^\lambda \otimes t_{-\lambda} \mapsto L^{\lambda+\mu} \otimes t_{-\lambda-\mu}$.
2. The collection $\{\mathrm{SDT}(\lambda) \otimes \mathcal{T}_{-\lambda}\}_{\lambda \in \Lambda^-}$ together with the inclusion maps above form a directed system.
3. The set $\mathrm{SDT}(-\infty)$ together with e_i, f_i defined above is an abstract $\mathfrak{q}(n)$-crystal such that
$$\mathrm{SDT}(-\infty) \cong \varinjlim_{\lambda \in \Lambda^-} \mathrm{SDT}(\lambda) \otimes \mathcal{T}_{-\lambda}.$$

Theorem (Salisbury–Scrimshaw)

Let $\lambda \in \Lambda^-$ such that $\lambda_i < \lambda_{i+1}$ for all $i \in I_0$. As $\mathfrak{q}(n)$-crystals using the modified tensor product rule, the connected component of $\mathrm{SDT}(-\infty) \otimes \mathcal{R}^\vee_{w_0\lambda}$ generated by $L^{-\infty} \otimes r^\vee_{w_0\lambda}$ is isomorphic to $\mathrm{SDT}(\lambda)$.

The $\mathfrak{q}(3)$-crystal $\mathrm{SDT}(\lambda)$ with $\lambda = -3\epsilon_1 - \epsilon_2$