Complexity, Combinatorial Positivity, and Newton Polytopes

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Introduction

The nonvanishing problem asks if a coefficient of a polynomial is nonzero. Many families of polynomials in algebraic combinatorics admit combinatorial counting rules and simultaneously enjoy having saturated Newton polytopes (SNP). Thereby, in amenable cases, nonvanishing is in the complexity class NP $\cap$ coNP of problems with "good characterizations". This suggests a new algebraic combinatorics viewpoint on complexity theory.

This paper focuses on the case of Schubert polynomials. These form a basis of all polynomials and appear in the study of cohomology rings of flag manifolds. We give a tableau criterion for nonvanishing, from which we deduce the first polynomial time algorithm. These results are obtained from new characterizations of the Schubitope, a generalization of the permutahedron defined for any subset of the $n \times n$ grid, together with a theorem of A. Fink, K. Mészáros, and A. St. Dizier (2018), which proved a conjecture of C. Monical, N. Tokcan, and A. Yong (2017).

Decision Problems

A **decision problem** is a problem with a yes or no answer given some input. Some problems have quick algorithms while others seem to require a lengthy search to reach an answer. To better understand these difference, problems are sorted into complexity classes.

Some complexity classes with examples:

- NP: LP ($\exists x \geq 0$, Ax=b?)
- coNP: Primes
- P: LP and Primes
- NP-complete: Graph coloring

Problem 1 *When do these coincide?*

- P $\stackrel{?}{=}$ NP
- NP $\stackrel{?}{=}$ coNP
- NP $\cap$ coNP $\stackrel{?}{=}$ P

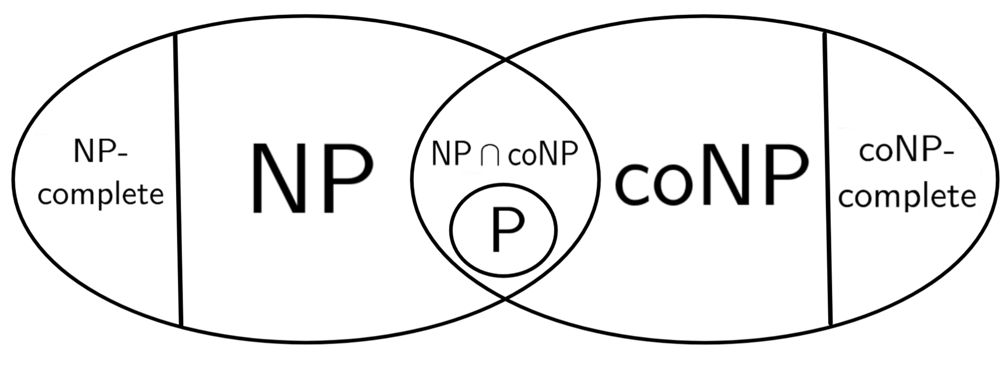


Figure 1: *Many believe the equalities in Problem 1 do not hold, giving the diagram above.*

Nonvanishing

In algebraic combinatorics we often study polynomial families:

$$F_\diamond = \sum_\alpha c_{\alpha,\diamond} x^\alpha = \sum_{s \in S} \mathrm{wt}(s) \in \mathbb{Z}[x_1, \ldots, x_n]$$

Example 2 *Below are a few different families:*

- *With $\diamond = \lambda$, use $F_\lambda = s_\lambda$, the Schur polynomial, and $S = \mathrm{SSYT}(\lambda)$. For instance, $s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2$ since*

$$\mathrm{SSYT}((2,1)) = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1} \end{array} \right\}.$$

- *With $\diamond = G$, use $F_G = \chi_G$, Stanley's chromatic symmetric polynomial, and $S = \{$proper colorings of $G\}$.*
- *With $\diamond = w \in S_\infty$, use $F_w = \mathfrak{S}_w$, but there are many choices for S.*

In this framework, we can discuss the complexity of the **nonvanishing problem**:

Problem 3 *What is the complexity of deciding $c_{\alpha,\diamondsuit} \neq 0$, as measured in the input size of α and $\diamondsuit$?*

In our cases of interest, $c_{\alpha,\diamondsuit} \in \mathbb{Z}_{\geq 0}$ has **combinatorial positivity**, which implies $\mathrm{nonvanishing}(F_\diamondsuit) \in$ NP.

Newton Polytopes

To $F_\diamond$, we can associate its **Newton polytope**:

$$\mathrm{Newton}(F_\diamond) = \mathrm{conv}\{\alpha : c_{\alpha,\diamond} \neq 0\} \subseteq \mathbb{R}^n$$

C. Monical-N. Tokcan-A. Yong '17 defined that $F_\diamond$ has **saturated Newton polytope** (SNP) if

$$\beta \in \mathrm{Newton}(F_\diamond) \iff c_{\beta,\diamond} \neq 0.$$

Example 4 *Let $f = x_1 x_2^3 + x_1^3 x_2^2 + x_1 x_2^2 + x_1 x_2$. Then f does not have SNP since $(2,2) \in \mathrm{Newton}(f)$ but $x_1^2 x_2^2$ does not appear in the monomial expansion of f.*

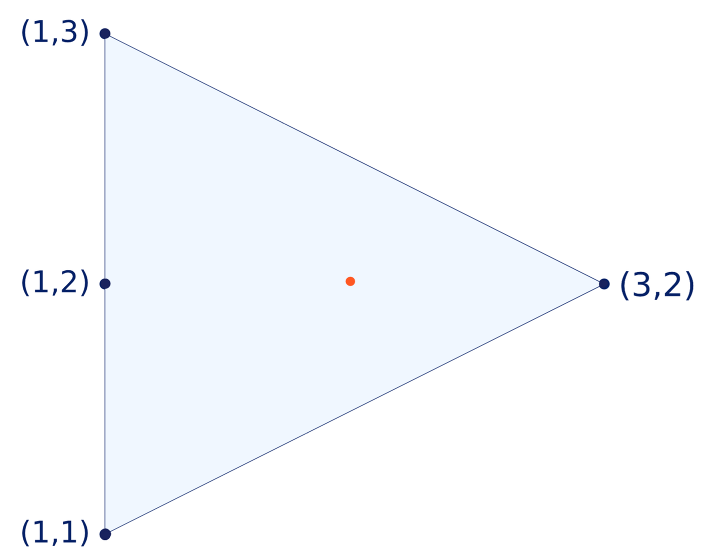


Figure 2: $\mathrm{Newton}(f)$ *of f in Example 4*

SNP combined with a polynomial-size halfspace description of $\mathrm{Newton}(F_\diamond)$ implies $\mathrm{nonvanishing}(F_\diamond) \in$ coNP. Therefore, in many cases $\mathrm{nonvanishing}(F_\diamond) \in$ NP $\cap$ coNP.

Example 5 *Below is an example and non-example of SNP:*

- *s_λ has SNP and* $\mathrm{nonvanishing}(s_\lambda) \in$ P
- *χ_G does not have SNP for G arbitrary, and* $\mathrm{nonvanishing}(\chi_G) \in$ NP. *In fact, for each fixed $n \geq 3$ it is* NP-complete.

If some $F_\diamond$ with SNP is such that $\mathrm{Nonvanishing}(F_\diamond)$ is NP − complete, then a polynomial-size halfspace description of $\mathrm{Newton}(F_\diamond)$ implies

$$\text{coNP} \cap \text{NP} - \text{complete} \neq \emptyset \implies \text{NP} = \text{coNP}.$$

Potential Application

Conjecture 6 (R. P. Stanley '95) *If G is claw-free (i.e., it contains no induced $K_{1,3}$ subgraph), then χ_G is Schur positive.*

Conjecture 7 (C. Monical '17) *If χ_G is Schur positive, then it is SNP.*

Combining these gives

Conjecture 8 (A. Adve-C. Robichaux-A.Yong, '18) *If G is claw-free then χ_G is SNP.*

I. Holyer proved n-coloring claw-free graphs is NP-complete. Therefore:

> An polynomial-size halfspace description proves $\mathrm{nonvanishing}(\chi_{\text{claw-free } G})$ is coNP. This implies NP = coNP.

Schubert Polynomials

Schubert polynomials form a linear basis of all polynomials $\mathbb{Z}[x_1, x_2, x_3, \ldots]$. They were introduced by A. Lascoux–M.-P. Schützenberger to study the cohomology ring of the flag manifold.

For $w_0 = n\ n-1\ \cdots 2\ 1 \in S_n$,

$$\mathfrak{S}_{w_0}(x_1, \ldots, x_n) := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

Otherwise, for $w \neq w_0$, apply Newton's divided difference operator

$$\partial_i f = \frac{f - f^{s_i}}{x_i - x_{i+1}},$$

recursively using weak Bruhat order to define $\mathfrak{S}_w$. To each $w \in S_\infty$ there is a unique **code**,

$$\mathrm{code}(w) = (c_1, c_2, \ldots, c_L) \in \mathbb{Z}_{\geq 0}^L,$$

where c_i counts the number of boxes in the i-th row of the Rothe diagram $D(w)$ of w.

Let Schubert be $\mathrm{nonvanishing}(\mathfrak{S}_w)$. The `INPUT` is $\mathrm{code}(w) = (c_1, \ldots c_L)$ with $c_L > 0$ and $\alpha \in \mathbb{Z}_{\geq 0}^L$.

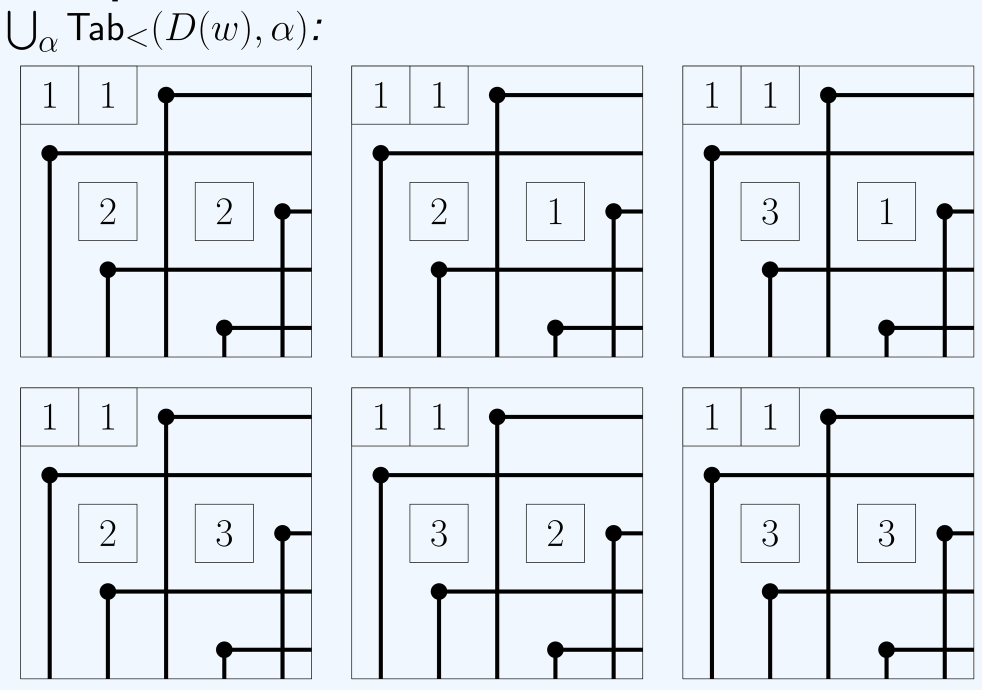
Theorem 9 (A. Adve-C. Robichaux-A.Yong, '18)
Schubert $\in$ P.

For $w \in S_n$, let $\mathrm{Tab}(w, \alpha)$ be the fillings of $D(w)$ with α_k many k's, where entries in each column are distinct, and any entry in row i is $\leq i$. We prove Theorem 9 using the following:

Theorem 10 (A. Adve-C. Robichaux-A.Yong, '18)
$c_{\alpha,w} \neq 0$ *if and only if* $\mathrm{Tab}(D(w), \alpha) \neq \emptyset$.

In general $\#\mathrm{Tab}(D(w), \alpha) \geq c_{\alpha,w}$.

Example 11 *For $w = 31524$, the tableaux in the set $\bigcup_\alpha \mathrm{Tab}_<(D(w), \alpha)$:*



Hence, for instance, $c_{(2,1,1),31524} > 0$ but $c_{(4),31524} = 0$.

Fix $n \in \mathbb{Z}_{>0}$ and let $D \subseteq [n]^2$. We call D a **diagram** and visualize D as a subset of an $n \times n$.

In 2017, C. Monical-N. Tokcan-A. Yong defined the **Schubitope** $\mathcal{S}_D$, a polytope defined for $D \subseteq [n]^2$, and conjectured the following:

Theorem 12 (A. Fink-K. Mészáros-A. St. Dizier, '17)
$\mathcal{S}_D = \mathrm{Newton}(\mathfrak{S}_w)$.

Our results give a polynomial time algorithm to check if a lattice point is in $\mathcal{S}_D$. This more general result gives a polynomial time algorithm for any polynomial family whose Newton polytopes are Schubitopes.

Additionally, we show that while the nonvanishing problem is easy, the counting problem is hard:

Theorem 13 (A. Adve-C. Robichaux-A.Yong, '18)
$c_{\alpha,w}$ *is* #P*-complete.*

Proof Sketch of Theorem 9

- By Theorem 12, $c_{\alpha,w} \neq 0$ if and only if $\alpha \in \mathcal{S}_D$.
- Prove $\alpha \in \mathcal{S}_D$ if and only if $\mathrm{Tab}(D, \alpha) \neq \emptyset$.
- Then introduce a new polytope $\mathcal{P}(D, \alpha)$ whose integer points biject with $\mathrm{Tab}(D, \alpha)$.
- Integer linear programming is hard, but $\mathcal{P}(D, \alpha)$ is totally unimodular. Now use LPfeasibility $\in$ P.

Conclusion

- We described an algebraic combinatorics paradigm for complexity on theoretical computer science.
- Conversely, complexity gives some new perspectives on algebraic combinatorics.
- We obtain new results about Schubert polynomials and the Schubitope.

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