Restricted Tutte polynomials for some periodic oriented forests on infinite square lattice

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Tree, forest and parenthood function

Let $G = (V, E)$ be a finite simple connected graph. We encode a rooted spanning tree as an endofunction R called *parenthood function* that maps for any vertex $v \in V$ its parent or itself for the root. The *ray* of $u \in V$ in R is the set of edges in the orbit $(u, R(u), R^2(u)\dots)$ of u in R: $\mathrm{Ray}(u) := (R^i(u)R^{i+1}(u))_{i\geq 0}$. An edge $uv \in E$ is *external* for R if $R(u) \neq v$ and $R(v) \neq u$. The *fundamental cycle* of this edge is the symmetrical difference of $\mathrm{Ray}(u)$ and $\mathrm{Ray}(v)$: $C_{uv}(R) = \mathrm{Ray}(u)\Delta\mathrm{Ray}(v)$. The *co-cycle* of an internal edge is the cut between the two sets of vertices when disconnecting the tree induced by R by removing this edge.

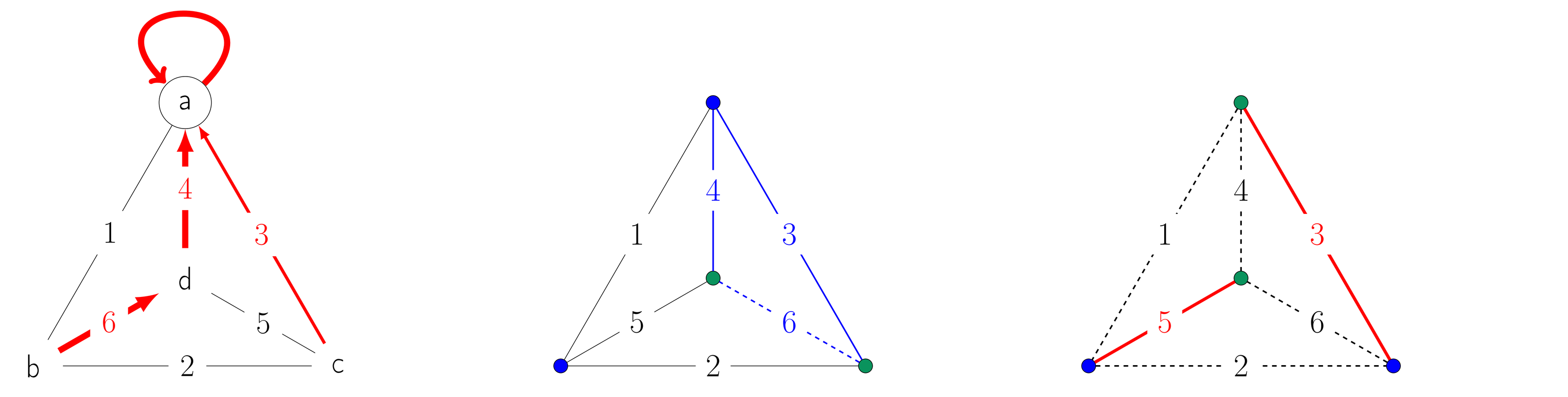


Figure 1. Example of weakly acyclic parenthood function on K_4 in red. Here $\mathrm{Ray}(b) = \{bd, da, aa\}$ (bold red). In middle, fundamental cycle of dc in blue. On the right, co-cycle of ad (black dots)

Tutte polynomial on finite graphs

Given an order on the edges (shown as numbers on Figure 1)

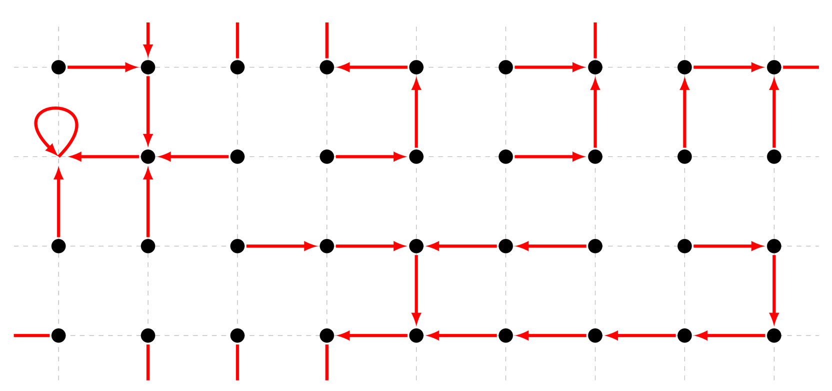
$$T_G(q,t) := \sum_T q^{\mathsf{ext}(T)} t^{\mathsf{int}(T)}$$

where T runs over the set $\mathcal{T}_G$ of spanning trees of G and $\mathsf{ext}(T)$, respectively $\mathsf{int}(T)$, is the external, respectively internal, (Tutte) activity. The external activity of a tree is the number of external edges that are minimal in their fundamental cycles. The internal activity of a tree is the number of internal edges that are minimal in their co-cycles.

On Figure 1, $\{ab, bc\}$ are external active edges then $\mathsf{ext}(T) = 2$ and there is no internal active edge, then $\mathsf{int}(T) = 0$. $T_{K_4}(q,t) = q^3 + t^3 + 3q^2 + 3t^2 + 4qt + 2q + 2t$

Lemma: (Tutte (1954)) For any order $<_E$, any elementary transposition $\tau_i = (i, i{+}1)$, the involution $\Phi_{\tau_i,<_E}$ maps a tree T of external activity k for order $<_E$ to a tree $\Phi_{\tau_i,<_E}(T)$ with the same external activity k for order $<_{\tau_i.E}$.

The square lattice



We aim for an analogue of the Tutte polynomial for the graph $\mathbb{Z}^2$. We need for summability and computability:

- Finite set of parenthood function.
- Finite activities

This suggests to use periodicity and leads to the following restrictions.

Restrictions

Periodic spanning forest: A spanning forest F of $\mathbb{Z}^2$ is *periodic of period* $(W, H) \in \mathbb{N}_+^2$ if for any edge $uv \in F$, the edge obtained by the translation of vector (kW, lH) is in F for any $(k, l) \in \mathbb{Z}^2$. It is *admissible* if there is no finite tree. (NB: this can't be a tree)

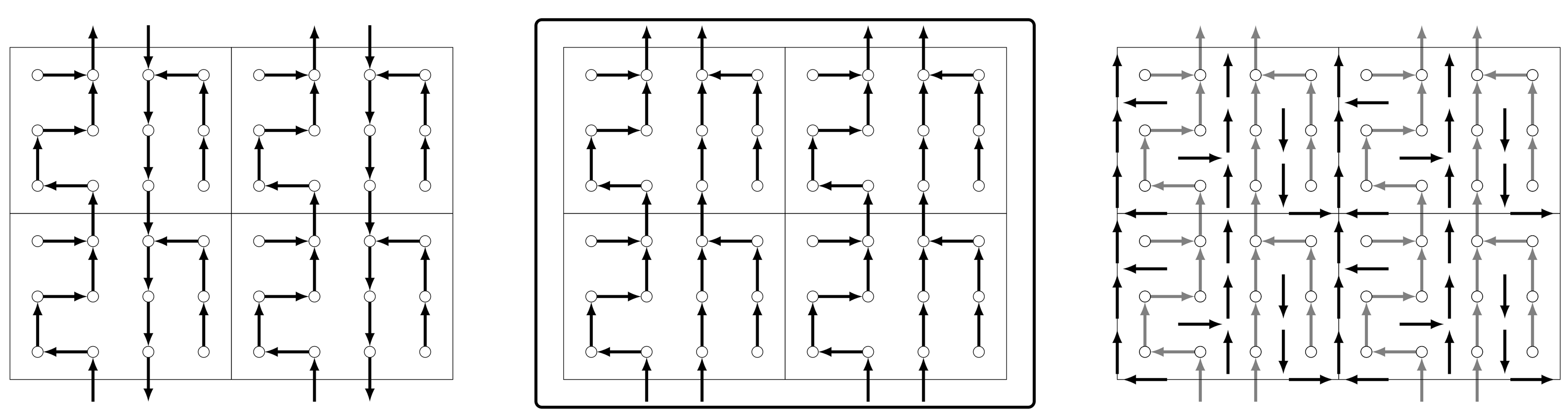


Figure 2. Examples of acyclic periodic parenthood function of period $(4, 3)$ (left: wrongly directed, center: directed towards $\frac{\pi}{2} = (0, 1)$, right: dual forest)

Activities on fundamental domains: We count the external activity on the (finite) set of edges in a fundamental domain. Since the graph is planar and self-dual, the internal activity of a forest is computed from the external activity of its dual.

Order, infinite paths and cycles, and activity

Let $\vec{\theta} \in \mathbb{Z}^2$ be a non null vector coding the direction of the root and $\vec{\theta}^{\perp} = (-\theta_y, \theta_x)$. We define a total order $<_{E_\theta}$ on edges of the square lattice:

$$e_i <_{E_\theta} e_j \iff (\langle \vec{e_i}, \vec{\theta}\rangle, \langle \vec{e_i}, \vec{\theta}^{\perp}\rangle) <_{\mathsf{lex}} (\langle \vec{e_j}, \vec{\theta}\rangle, \langle \vec{e_j}, \vec{\theta}^{\perp}\rangle)$$

We encode an admissible periodic spanning forest by an *acyclic* parenthood function without fixed point. Infinite paths are oriented toward $\vec{\theta}$ expect if orthogonal then oriented toward $\vec{\theta}^{\perp}$. Fundamental cycles can be infinite.

Lemma: For any external edge uv of a periodic spanning forest F of period (W, H), the minimal edge of its fundamental cycle is among the $2WH$ first edges of $Ray_F(u)$ or $Ray_F(v)$.

Results

Definition: Bivariate restricted Tutte polynomial

$$T_{\vec{\theta},W\times H}(q,t) := T_{\mathcal{F}^{W\times H},\mathcal{E}_{W\times H},<_{E_\theta}}(q,t) = \sum_{F\in\mathcal{F}^{W\times H}} q^{\mathsf{ext}_{\mathcal{E}_{W\times H}}(F)} t^{\mathsf{int}_{\mathcal{E}_{W\times H}}(F)}$$

We sum on the admissible periodic spanning forests of period (W, H).

Properties:

- **Decidable:** it can be computed in finite time
- **Symmetry:** By design this polynomial is symmetric in q and t: $T_{\vec{\theta},W\times H}(q,t) = T_{\vec{\theta},W\times H}(t,q)$

Theorem (Derycke, Le Borgne 2018)

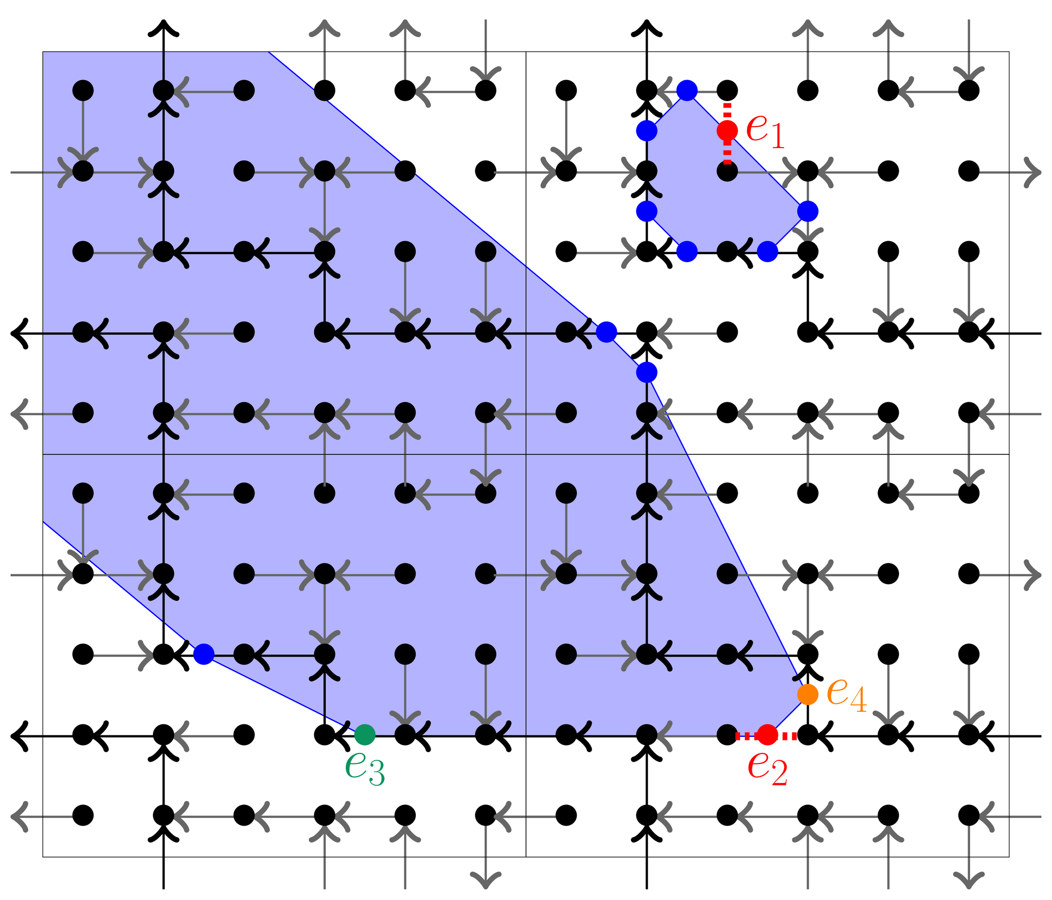
For any two rational directions $\vec{\theta}$ and $\vec{\theta}'$ we have $T_{\vec{\theta},W\times H}(q,1) = T_{\vec{\theta}',W\times H}(q,1)$.

But $T_{\vec{\theta},W\times H}(q,t) \neq T_{\vec{\theta}',W\times H}(q,t)$ in general:

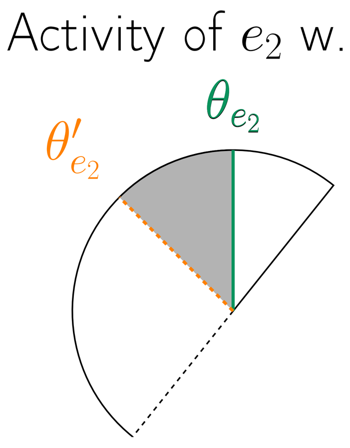
$T_{(0,1),3\times 1}(q,t) = q^3t^3 + 3qt^2 + 3q^2t + 3q + 3t + 4,$

$T_{(-1,0),3\times 1}(q,t) = q^3t^3 + 3q^2 + 3t^2 + 3qt + 3q + 3t + 1.$

Key ideas of the Theorem



Activity of e_2 w.r.t. $\vec{\theta}$



- Rotate $\vec{\theta}$ counter-clockwise.
- Exchange (periodicaly) e_2 and e_3 if $\vec{\theta} = \vec{\theta}_{e_2}$ or e_2 and e_4 if $\vec{\theta} = \vec{\theta}'_{e_2}$. (infinite version of Tutte lemma that may create or destroy infinite cycles)

Application for the sandpile model

Weak Dhar Criterion: Weak version of the Dhar criterion characterizing "recurrent" configurations on $\mathbf{Z}^2$ in the direction $\vec{\theta}$ by toppling a half-plane orthogonal to $\vec{\theta}$.

Distribution of grains: $T_{\vec{\theta},W\times H}(q,1)$ counts the distribution of grains on the periodic "recurrent" configurations of period (W, H) satisfying the previous criterion.

[1] W. T. Tutte. *A Contribution to the Theory of Chromatic Polynomials.* Cambridge University Press, V.6 80–91 (1954).

[2] H. Bruhn and R. Diestel. *Infinite matroids in graphs.* Discrete Mathematics, V.311:15 1461–1471 (2011).

[3] D. Dhar. *Theoretical studies of self-organized criticality.* Physica A Statistical Mechanics and its Applications, V.369 29–70 (2006).

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