On the Schur positivity of sums of power sums

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The problem and two examples

Let T be a nonempty subset of positive integers and p_n the nth power sum symmetric function. Consider the multiplicity-free sum of power sums

$$F_n^T = \sum_{\lambda \vdash n} p_\lambda,$$

the sum ranging over partitions of n with parts λ_i in T. Note F_n^T is the degree n term in

$$\prod_{n \in T} (1 - p_n)^{-1}.$$

Question 1: For what subsets T is F_n^T **Schur-positive**, that is, for which T can F_n^T be expanded in the Schur basis with only **nonnegative** coefficients?

Question 2: More generally, what subsets U of partitions yield a Schur positive sum of power sums $\sum_{\lambda \in U} p_\lambda$?

Example 1: Let $T = \{1\}$. Then $F_n^T = p_1^n$, so F_n^T is Schur-positive by the well-known decomposition of the regular representation of S_n :

$$F_n^T = p_1^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda.$$

Example 2: Let $T = \mathbb{N}^+$ (all positive integers). Then

$$F_n^T = \sum_{\lambda \vdash n} p_\lambda.$$

Theorem (L. Solomon, 1961): $F_n^{\mathbb{N}^+}$ is the Frobenius characteristic of S_n acting on itself by conjugation, hence it is Schur-positive. (Coefficients are unknown.)

A new symmetric function

Fix a nonempty subset T of the positive integers. Define a function ψ^T on the set of positive integers by

$$\psi^T(d) = \sum_{m|d,\, m \in T} m\, \mu\left(\frac{d}{m}\right),$$

where $\mu(d)$ is the Möbius function.

Also define a sequence of (possibly virtual) representations indexed by the subset T, with Frobenius characteristic

$$f_n^T = \frac{1}{n} \sum_{d|n} \psi^T(d) p_d^{\frac{n}{d}}.$$

Set $F^T = \sum_{n \geq 1} f_n^T$, $\quad p^T = \sum_{n \in T} p_n$.

Example 1. $T = \{1\} \implies \psi^T(d) = \mu(d)$

$$\implies f_n^T = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{\frac{n}{d}}$$

is Lie_n, the S_n-action on the multilinear component of the free Lie algebra with n generators.

Example 2. $T = \mathbb{N}^+ \implies \psi^T(d) = \phi(d)$ (Euler phi function)

$$\implies f_n^T = \frac{1}{n} \sum_{d|n} \phi(d) p_d^{\frac{n}{d}}$$

is $Conj_n$, the conjugacy action of S_n on the n-cycles.

A formula for $\sum_{\lambda \in Par,\, \lambda_i \in T} p_\lambda$.

Let H (resp. E) be the sum of the homogeneous (resp. elementary) symmetric functions $\{h_n\}_{n \geq 0}$ (resp. $\{e_n\}_{n \geq 0}$). Let Lie denote the sum of the symmetric functions $\{Lie_n\}_{n \geq 1}$. Then, with $[\,,\,]$ denoting plethysm:

Main Theorem (S, 2018):

$$H[F^T] = \prod_{n \in T} (1 - p_n)^{-1} = \sum_{\lambda \in Par,\, \lambda_i \in T} p_\lambda. \qquad (1)$$

$$F^T = p^T[Lie] = \sum_{m \in T} Lie[p_m], \qquad (2)$$

$$f_n^T = \sum_{m \in T,\, m|n} Lie_{\frac{n}{m}}[p_m]. \qquad (3)$$

If $G^T = \sum_{k \geq 0} \sum_{m \in T} Lie[p_{m \cdot 2^k}]$, then

$$E[G^T] = \sum_{\lambda \in Par,\, \lambda_i \in T} p_\lambda = H[F^T]. \qquad (4)$$

Corollary: If F^T or G^T is Schur-positive, then so is

$$\sum_{\lambda \in Par,\, \lambda_i \in T} p_\lambda.$$

The cases $T = \{1\}$ and $T = \mathbb{N}^+$

Eqn. (1) implies the following classical results, and hence Schur positivity of the corresponding sums of power sums:

$$T = \{1\} : \text{(Thrall, 1942)} \quad \sum_{r \geq 1} h_r[Lie] = \sum_{n \geq 1} p_1^n. \qquad (5)$$

$$T = \mathbb{N}^+ : \text{(Solomon, 1961)} \quad \sum_{r \geq 1} h_r[Conj] = \sum_{\lambda \in Par} p_\lambda. \qquad (6)$$

Subsets of primes

Let $S = \{q_1, \ldots, q_k, \ldots\}$ be a set of distinct primes. Factor n uniquely into $n = Q_n \ell_n$ where $Q_n = \prod_{q \in S} q^{a_q(n)}$ for nonnegative integers $a_q(n)$, and $(\ell_n, q) = 1$ for all $q \in S$. For each $n \geq 1$, define

$$L_n^S = \frac{1}{n} \sum_{d|n} \psi(d) p_d^{\frac{n}{d}} \quad \text{with } \psi(d) = \phi(Q_d)\mu(\ell_d), \qquad (7)$$

Theorem: The symmetric function L_n^S is the Frobenius characteristic of a true S_n-module, and is thus Schur-positive.

Theorem: Let S be a set of primes, and let $P(S)$ be the set of positive integers whose prime divisors are a subset of S (so $1 \in P(S)$). Then

$$H[L^S] = \prod_{n \in P(S)} (1 - p_n)^{-1} = \sum_{\lambda \in Par,\, \lambda_i \in P(S)} \mathbf{p}_\lambda, \qquad (8)$$

and hence this sum is Schur-positive.

The case $T = \{q^r : r \geq 0\}$, q prime

Note: $Lie_n = L_n^\emptyset$, $Conj_n = L_n^{\mathbb{N}^+}$. Set $Lie_n^{(q)} = L_n^{\{q\}}$.

Corollary: Let $S = \{q\}$ consist of a single prime q. The following sums of power sums are Schur-positive:

$$\sum_{\substack{\lambda \in Par \\ \lambda_i \text{ a power of } q}} \mathbf{p}_\lambda \quad \text{and} \quad \sum_{\substack{\lambda \in Par \\ (\lambda_i, q) = 1}} \mathbf{p}_\lambda$$

The special case $q = 2$ has unique properties:

Theorem:

$$\sum_{\lambda : \lambda_i \equiv 1 \bmod 2} \mathbf{p}_\lambda = \prod_{n \equiv 1 \bmod 2} (1 - p_n)^{-1} = E[Conj] = H[L^{\overline{(2)}}]. \qquad (9)$$

The case $T = \{k^r : r \geq 0\}$, $k \geq 2$

Theorem:

$$H[\sum_{n \geq 1} f_n^T] = \prod_{r \geq 0} (1 - p_{k^r})^{-1},$$

$$f_n^T = \begin{cases} Lie_n + f_{\frac{n}{k}}^T[p_k], & k|n; \\ Lie_n, & \text{otherwise}, \end{cases} \qquad (10)$$

When k is prime, this is Schur-positive.

For $k = 4$, f_n^T is not Schur-positive when $n = 4, 16$, and the degree 16 term in the product $\prod_{r \geq 0} (1 - p_{4^r})^{-1}$ is not Schur-positive. In both cases it is only the sign representation that appears with coefficient (-1).

Conjecture 1: For all ODD positive integers k, f_n^T is Schur-positive, and hence so is $\prod_{r \geq 0} (1 - p_{k^r})^{-1}$.

The case $T = \{1, k\}$

$$f_n^T = \begin{cases} Lie_n + Lie_{\frac{n}{k}}[p_k], & k|n; \\ Lie_n, & \text{otherwise}, \end{cases} \qquad (11)$$

and $(1 - p_1)^{-1}(1 - p_k)^{-1} = H[\sum_{n \geq 0} f_n^T]$.

Theorem: The sum $\sum_{\lambda : \lambda_i = 1, k} p_\lambda$ is Schur-positive. If k is prime, then $f_n^{\{1,k\}} = \mathrm{ch}(\exp \frac{2ki\pi}{n}) \uparrow_{C_n}^{S_n}$ is Schur-positive.

Conjecture 2: For all ODD positive integers k, $f_n^{\{1,k\}}$ is Schur-positive.

The case $T = \{n : n|k\}$, $k \geq 2$

Theorem:

$$f_n^T = \sum_{m|(k,n)} Lie_{\frac{n}{m}}[p_m], \quad \prod_{k \equiv 0 \bmod n} (1 - p_n)^{-1} = H[\sum_n f_n^T].$$

Then $f_n^T = \mathrm{ch}(\exp \frac{2ki\pi}{n}) \uparrow_{C_n}^{S_n}$ is Schur-positive, and also $\sum_{\lambda : \lambda_i | k} \mathbf{p}_\lambda$.

$T = \{n : n \equiv 1 \bmod k\}$, $k \geq 3$

Theorem: (See Eqn. (9) for $\mathbf{k = 2}$)

$$f_n^T = \sum_{\substack{m \equiv 1 \bmod k \\ m|n}} Lie_{\frac{n}{m}}[p_m], \quad \prod_{n \equiv 1 \bmod k} (1 - p_n)^{-1} = H[\sum_n f_n^T].$$

Conjecture 3: $\prod_{n \equiv 1 \bmod k} (1 - p_n)^{-1}$ is Schur-positive (verified for $n \leq 24$, $k \leq 6$).

The case $T = \{n : n \leq k\}$

Theorem:

$$f_n^T = \sum_{\substack{m=1 \\ m|n}}^{k} Lie_{\frac{n}{m}}[p_m], \quad \prod_{n=1}^{k} (1 - p_n)^{-1} = H[\sum_n f_n^T].$$

If **n is prime**, or **$n \leq k$**, or **$n > k$ with greatest proper divisor at most k**, then f_n^T is Schur-positive.

Conjecture 4: $f_n^{\{1,\ldots,k\}}$ is Schur-positive for all n and k, and hence so is $\prod_{n=1}^{k} (1 - p_n)^{-1}$.

New Plethystic Identities

Theorem:

$$\sum_{m \geq 1} p_m[Lie] = \sum_{n \geq 1} Conj_n; \qquad (12)$$

Let q be prime, and let $n = \ell q^k$ where $(\ell, q) = 1$.

$$Lie_n^{(q)} = \sum_{r=0}^{k} Lie_{\ell q^{k-r}}[p_{q^r}]. \qquad (13)$$

Reverse Lexicographic Order

This is a total order on the set of partitions of n : λ is greater than μ if $\lambda_1 > \mu_1$ or there is an index $j \geq 2$ such that $\lambda_i = \mu_i$ for $i < j$ and $\lambda_j > \mu_j$.

$(5) > (4,1) > (3,2) > (3,1^2) > (2^2,1) > (2,1^3) > (1^5)$

Conjecture 7 (S, 2016): The sum

$$\psi_\lambda = \sum_{\mu \vdash n,\, \lambda \geq \mu \geq (1^n)} p_\mu \qquad (14)$$

is Schur-positive for any partition λ of n.

Theorem: ψ_μ is Schur-positive if $\mu \leq (\mathbf{3}, \mathbf{1^{n-3}})$ or $\mu \geq (\mathbf{n-4}, \mathbf{1^4})$ in reverse lexicographic order, and also if $\mu = (\mathbf{3}, \mathbf{2^k}, \mathbf{1^r})$ for $k \geq 1$ and $0 \leq r \leq 2$.

Asymptotics

Let $f(n)$ be the number of nonempty subsets U of the $p(n)$ partitions of n, such that U **contains** (1^n) and the sum $\psi_{\mathbf{U}} = \sum_{\mu \in \mathbf{U}} p_\mu$ is **NOT** Schur-positive.

n	4	5	6	7	8	9	10
$p(n)$	5	7	11	15	22	30	42
$\mathbf{f(n)}$	**1**	**7**	**184**	**3674**	**488,259**	**145,796,658**	**670,141,990,673**
$\frac{f(n)}{2^{p(n)-1}}$	0.06	0.11	0.18	0.22	0.23	0.272	0.305

Conjecture 8: For $n \geq 6$, the numbers $\frac{f(n)}{2^{p(n)-1}}$ are increasing, and lie between $\frac{1}{16}$ and $\frac{1}{2}$. This would imply:

Conjecture 9 (Richard Stanley, 2018):

$\lim_{n \to \infty} \frac{f(n)}{2^{p(n)-1}}$ exists and is nonzero.