Simple recurrence formulas for bipartite maps with prescribed degrees

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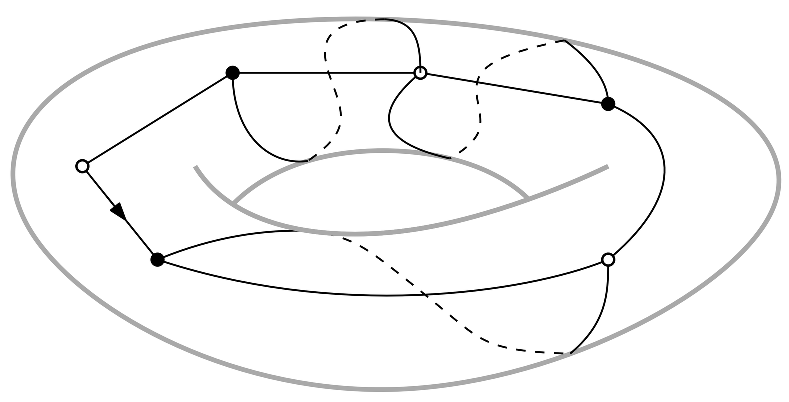
Introduction: maps

A map M is the embedding of a connected multigraph G in a (compact orientable) surface S, up to orientation-preserving homeomorphism. The faces of M are the connected components of $S \setminus G$, they must be homeomorphic to disks. The genus g of M is the genus of S (the number of handles). We will consider rooted maps: an oriented edge is distinguished.

A bipartite map is a map with black and white vertices and no monochromatic edges.



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Context and main result

Quadratic recurrence formulas for some *isolated* models of maps maps were derived from the **KP hierarchy** (an infinite set of PDEs coming from mathematical physics, see [2,4]). What about their *universality* ?

Using the (closely-related) **2-Toda hierarchy**, we obtain a formula for bipartite maps with *arbitrary* prescribed degrees.

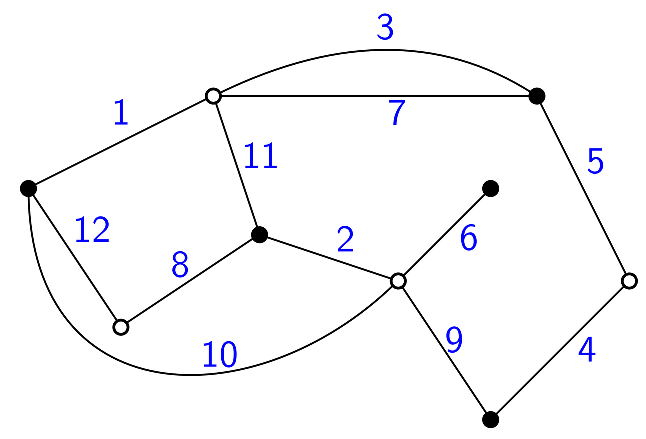
The number $B_g(\mathbf{f})$ of bipartite maps of genus g with f_i faces of degree $2i$ (for $\mathbf{f} = (f_1, f_2, \dots)$) satisfies:

$$\binom{n+1}{2} B_g(\mathbf{f}) = \sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ \mathbf{s},\mathbf{t}\neq 0\\ g_1+g_2+g^*=g}} (1+n_1)\binom{v_2}{2g^*+2} B_{g_1}(\mathbf{s})B_{g_2}(\mathbf{t}) + \sum_{g^*\geq 0}\binom{v+2g^*}{2g^*+2} B_{g-g^*}(\mathbf{f})$$

where $n = \sum_i i f_i$, $n_1 = \sum_i i s_i$, $v = 2-2g+n-\sum_i f_i$, $v_2 = 2-2g_2+n_2-\sum_i t_i$ and $n_2 = \sum_i i t_i$ (the n's count edges, the v's count vertices, in accordance with the Euler formula).

Remark: This formula is the fastest way known of computing the coefficients $B_g(\mathbf{f})$.

Maps as factorizations of permutations and a generating function



$\sigma_\circ = (1,3,7,11)(2,6,9,10)(4,5)(8,12)$

$\sigma_\bullet = (1,12,10)(2,8,11)(3,5,7)(4,9)(6)$

$\phi = (1,8)(2,12)(3,4,10)(5,11,6,9)(7)$

Each vertex/face is a cycle (degree=size of cycle)

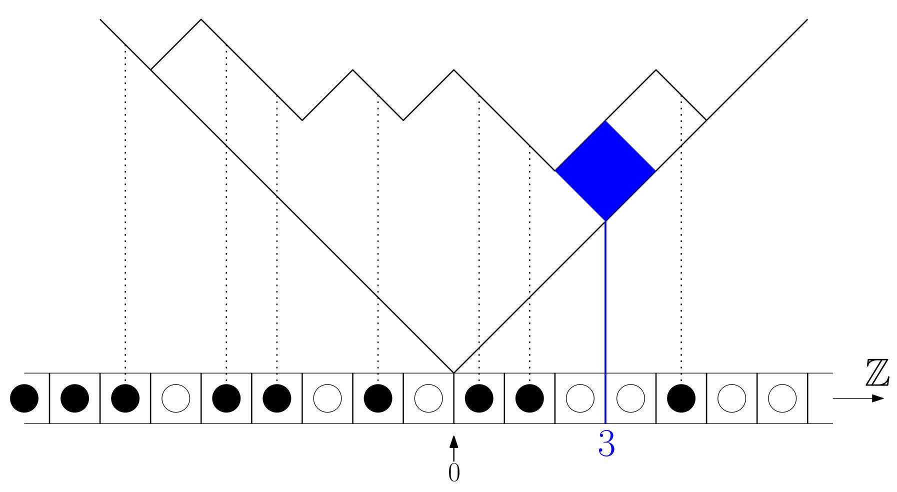
$$\sigma_\circ \sigma_\bullet = \phi$$

$$\tau(z,\mathbf{p},\mathbf{q},u) = \sum_{\substack{n\geq 0\\ |\mu|=|\lambda|=n\\ l\geq 2}} \frac{z^n}{n!} u^{2n-l} p_\lambda q_\mu W(l,\lambda,\mu)$$

$W(l,\lambda,\mu) = \#$ of 4-uples of permutations $(\sigma_1,\sigma_2,\sigma_\lambda,\sigma_\mu)$ of $\mathfrak{S}_n$ s.t. $\sigma_1\sigma_2\sigma_\lambda\sigma_\mu = 1$, (σ_1,σ_2) have l cycles in total, and $\sigma_\lambda, \sigma_\mu$ have respective cycle types λ and μ.

If we set $q_i = \mathbb{1}_{i=1}$, $\log\tau$ is the GF of bipartite maps with prescribed face degrees.

Maya diagrams and partitions



Maya diagram = decoration of $\mathbb{Z}+\frac{1}{2}$ by particles (black) and antiparticles (white) s.t. finite number of negative antiparticles, finite number of positive particles.

"Balanced" diagrams = partitions.

$\Lambda^{\frac{\infty}{2}}$= vector space whose orthonormal basis is the set of Maya diagrams.

Fermion operators: ψ_k/ψ_k^*: add/remove a particle in position k (up to a sign) if possible, otherwise 0.

Boson operators: $\alpha_n = \sum_k \psi_{k-n}\psi_k^*$ (for $n \in \mathbb{Z}\setminus\{0\}$.

Remark: $\alpha_{-n}|\lambda\rangle = \sum_{\mu,\nu}(-1)^\bullet|\mu\rangle$ for all ν border strips of size n and $\nu = \mu/\lambda$.

Vertex operators: $\Gamma_\pm(\mathbf{p}) = \exp(\sum_{n=1}^\infty \frac{p_n}{n}\alpha_{\pm n})$.

Remark: $\Gamma_-(\mathbf{p})|\emptyset\rangle = \sum_\lambda s_\lambda(\mathbf{p})|\lambda\rangle$.

The 2-Toda hierarchy

Casimir operator: $\Omega = \sum_k \psi_k \bigotimes \psi_k^*$.

Solutions of the 2-Toda hierarchy = functions of the form

$$\langle\emptyset|\Gamma_+(\mathbf{p})A\Gamma_-(\mathbf{q})|\emptyset\rangle$$

with $[A\bigotimes A, \Omega] = 0$.

τ can be expressed this way ! (see [4,5])

$$\frac{\partial^2}{\partial p_1 \partial q_1}\log\tau = \frac{\tau_1\tau_{-1}}{\tau^2} \qquad (1)$$

Outline of the proof

- Express the auxiliary functions $\tau_{\pm 1}$ in terms of τ, and transform (1) into a quadratic equation in $\log\tau$ (algebraic tricks),
- Interpret $\frac{\partial^2}{\partial p_1\partial q_1}$ combinatorially,
- Extract coefficients.

Comments and perspectives

- We also have formulas for other objects, namely constellations and monotone Hurwitz numbers.
- Our formula will be useful in studying the local limit of high genus maps with prescribed face degrees (in continuation of [1]).

References

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