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A bijection for Shi arrangement faces

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$$\#\left\{\begin{array}{c}\text{Catalan faces in } \mathbb{R}^n\\ \text{of dimension } k\end{array}\right\} = \sum_{i=k}^{n} S(n,i)(i-1)!\binom{i}{k}\binom{i+k}{k-1}$$

new bijective proof!

$$\#\left\{\begin{array}{c}\text{Shi faces in } \mathbb{R}^n\\ \text{of dimension } k\end{array}\right\} = \binom{n}{k}\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}(n-i+1)^{n-1}$$

new bijective proof!

Introduction

A *hyperplane arrangement* is a finite collection of affine hyperplanes in $\mathbb{R}^n$ for some $n \geq 1$. A *face* of a hyperplane arrangement is the solution set to a non-void system of equalities and inequalities, one for each hyperplane. The *dimension* of a face is the dimension of its affine span. We consider the combinatorial question of counting the faces of each dimension in the Catalan and Shi arrangements (right).

Background

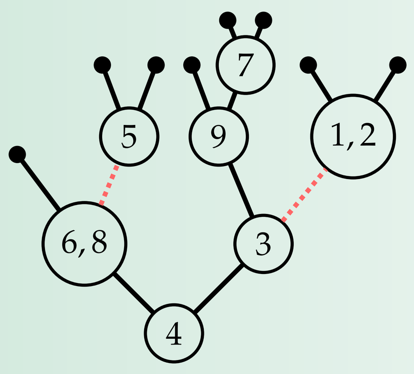
In 1996, Athanasiadis applied an intricate finite field method to obtain counting formulae for the faces of the Catalan and Shi arrangements [1]. He raised the question: is there a bijective explanation? By generalizing a bijection on regions defined by Olivier Bernardi in [2], we obtain an answer to this 23-year-old question.

Results Summary

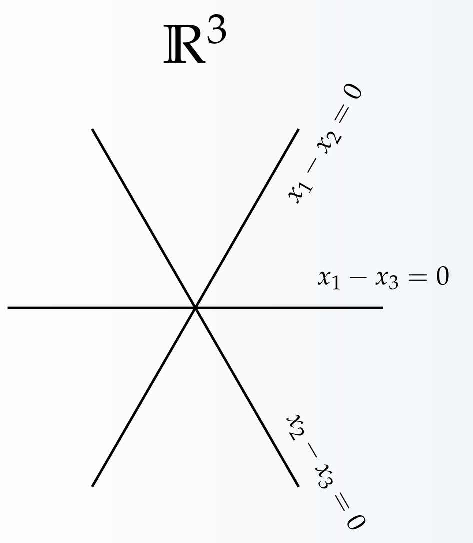
- We present an explicit bijection between the faces of the Catalan arrangement and certain decorated binary trees.
- We modify the above bijection to obtain an explicit bijection between the faces of the Shi arrangement and a simple subset of decorated binary trees (those trees without "right-ascents").
- We compose the above bijection with additional manipulations, to obtain an explicit bijection between the faces of the Shi arrangement and a simple set of marked sequences.

Trees

An *$[n]$-decorated binary tree* is a binary tree whose nodes are labelled by a partition of $[n]$, together with a marked subset of right internal edges (dashed).



Braid Arrangement $\quad x_i - x_j = 0 \quad$ for all $1 \leq i < j \leq n$



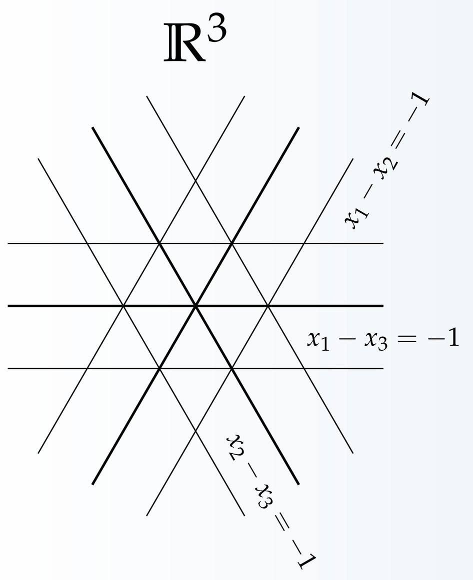
$\#\{\text{Braid regions}\} = n!$

Braid regions $\cong$ Permutations

$\#\{\text{Braid faces of dimension } k\} = k!S(n,k)$

Braid faces $\cong$ Ordered set partitions

Catalan Arrangement $\quad x_i - x_j = -1, 0, 1 \quad$ for all $1 \leq i < j \leq n$



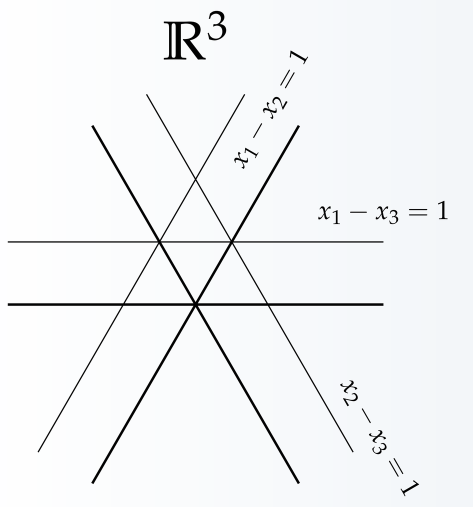
$\#\{\text{Catalan regions}\} = n!\text{Cat}_n$

Catalan regions $\cong$ Binary trees, etc!

$\#\{\text{Catalan faces of dimension } k\} =$ **see headline**

Catalan faces $\cong$ **???**

Shi Arrangement $\quad x_i - x_j = 0, 1 \quad$ for all $1 \leq i < j \leq n$



$\#\{\text{Shi regions}\} = (n+1)^{n-1}$

Shi regions $\cong$ Parking functions, Cayley trees

$\#\{\text{Shi faces of dimension } k\} =$ **see headline**

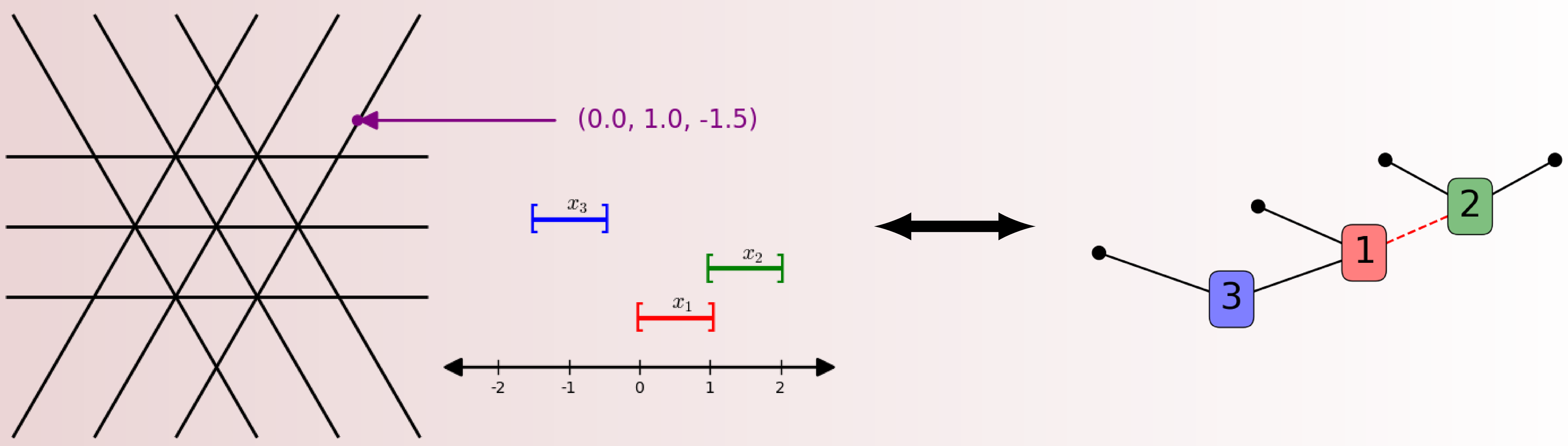
Shi faces $\cong$ **???**

Catalan Bijection $\quad$ Example with $n = 3$ and $k = 2$

We found a bijection between faces of the Catalan arrangement in $\mathbb{R}^n$ and $[n]$-decorated binary trees.

Furthermore, the k-dimensional faces correspond to trees with k solid right edges.

We can count $[n]$-decorated binary trees with Lagrange Inversion.

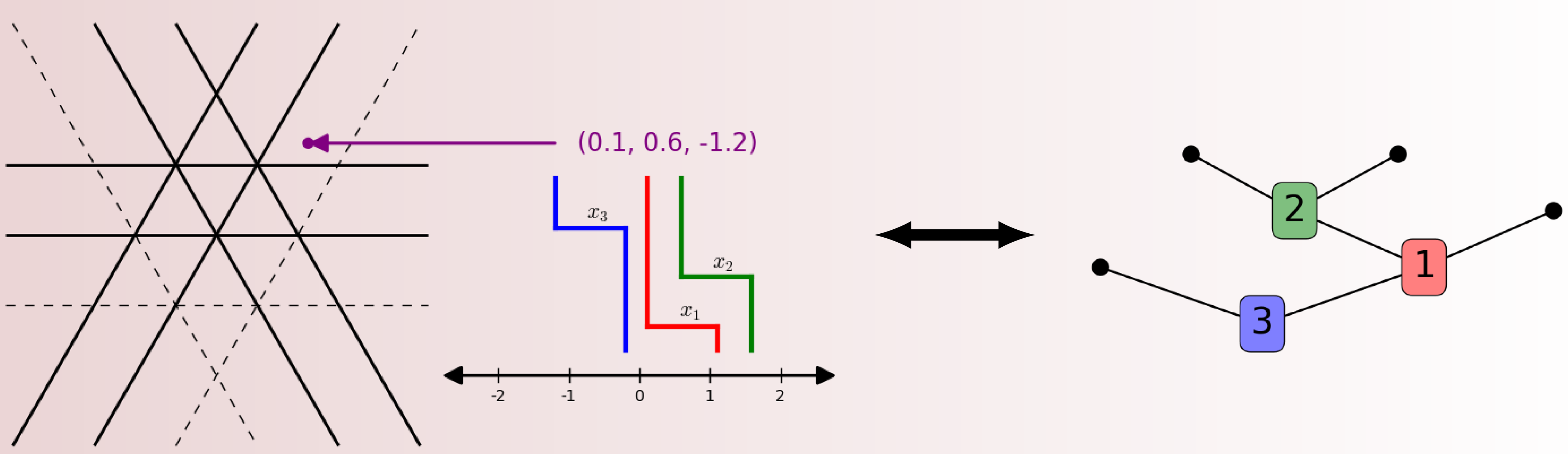


Shi Bijection $\quad$ Example with $n = 3$ and $k = 3$

A right internal edge with parent node P and child C is called a *right-ascent* if

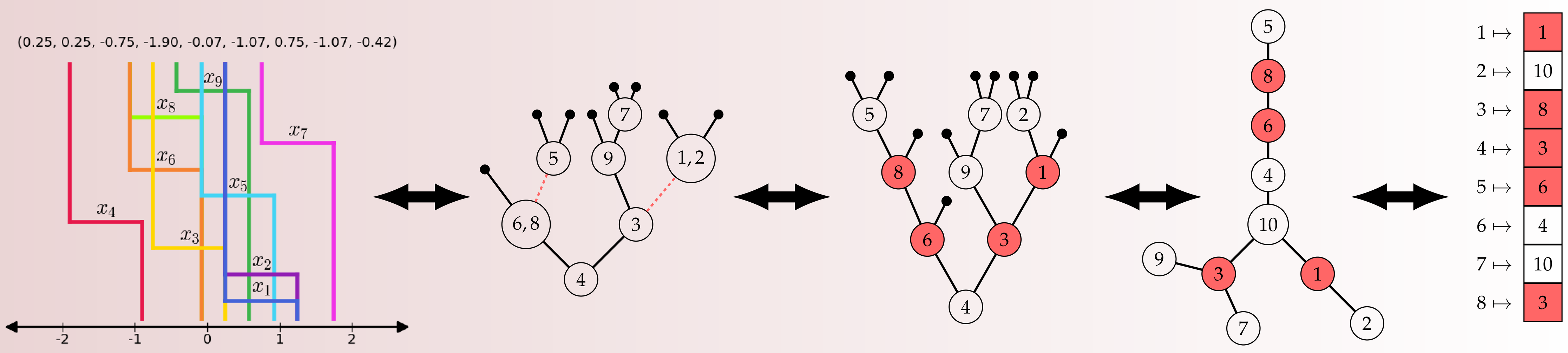
$$\max P < \min C.$$

We found a bijection between faces of the Shi arrangement in $\mathbb{R}^n$ and $[n]$-decorated binary trees with no right-ascents. Furthermore, the k-dimensional faces correspond to trees with k solid right edges. The basic idea is to show that for each Shi face, there is a unique representative among the Catalan faces partitioning it, whose corresponding tree has no right-ascents.



Shi Sequences $\quad$ Example with $n = 9$ and $k = 5$

A *Shi sequence* is a function $f : [n-1] \to [n+1]$, with a marked subset $S \subset \text{Im}(f) \setminus \{n+1\}$. We found a bijection between faces of the Shi arrangement and Shi sequences. Furthermore, the faces of dimension k correspond to Shi sequences with $|S| = n - k$. It is easy to count these Shi sequences by inclusion-exclusion.



Future Work

- Can similar results be obtained for the faces of the Linial arrangement?
- What additional face structure can be studied through this bijection (e.g. partial order, bounded/unbounded, Tits product)?

References

[1] C.A. Athanasiadis. "Algebraic combinatorics of graph spectra, subspace arrangements, and Tutte polynomials". PhD thesis. MIT, 1996.

[2] O. Bernardi. "Deformations of the braid arrangement and trees". In: *Advances in Mathematics* 335 (2018), pp. 466-518.