Resolving Stanley's conjecture on k-fold acyclic complexes

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Preliminaries

Definitions

A **simplicial complex** on n vertices is a subset Δ of $2^{[n]}$ such that

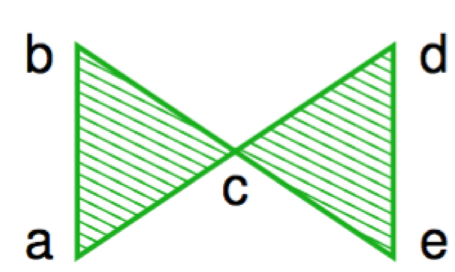
$$\sigma \in \Delta, \tau \subseteq \sigma \implies \tau \in \Delta.$$

Face: element of Δ; **Facet**: maximal face; $\dim \sigma = |\sigma| - 1$.

f-polynomial: $f(\Delta, t) = \sum_{\sigma \in \Delta} t^{|\sigma|} = f_{-1} + f_0 t + \cdots + f_{d-1} t^d$ where f_i is the number of faces of Δ of dimension i.

Link: $\operatorname{link} \sigma = \{\tau \in \Delta : \tau \cup \sigma \in \Delta, \tau \cap \sigma = \varnothing\}$

Example : $X = \langle abc, cde \rangle$, $f(X, t) = 1 + 5t + 6t^2 + 2t^3 = (1 + t)(1 + 4t + 2t^2)$.



A pure complex is **stacked** if its facets can be ordered $F_1, \ldots, F_j$ such that $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ is a simplex of dimension one less for all $i \in [j]$.

Simplicial Homology

- $\tilde{H}_i(\Delta, \Bbbk)$: i^{th} **reduced simplicial homology group** of Δ with coefficients in $\Bbbk$.
- $\tilde{\beta}_i = \dim_{\Bbbk} \tilde{H}_i(\Delta, \Bbbk)$: **reduced Betti numbers** of Δ. Can be thought of as counting the i-dimensional "holes" in Δ.
- Δ is **acyclic** (over $\Bbbk$) if $\tilde{\beta}_i = 0$ for all i. Acyclicity is topological (up to $\Bbbk$).

Known results

Theorem (Kalai, 1985)

If Δ is acylic over some field, then $f(\Delta, t) = (1 + t) f(\Delta', t)$ for some simplicial complex Δ'.

Kalai's result does not give an explicit connection between Δ and the complex Δ'.

Theorem (Stanley, 1993)

If Δ is acylic over some field, then Δ can be decomposed into a disjoint union of rank 1 boolean intervals whose minimal faces form a subcomplex of Δ.

This subcomplex realizes the Δ' expected by Kalai's result.

k-fold acyclicty

- A complex Δ is **k-fold acyclic** if $\operatorname{link} \sigma$ is acyclic for all $\sigma \in \Delta$ such that $|\sigma| < k$.
- Notice that 1-fold acyclic is equivalent to acyclic, but k-fold acyclic is not toplogical for $k > 1$.

The conjecture

Theorem (Stanley, 1993)

If Δ is k-fold acylic over some field, then $f(\Delta, t) = (1 + t)^k f(\Delta', t)$ for some complex Δ'.

Following the natural parallel from before, Stanley made the following conjecture.

Conjecture (Stanley, 1993)

If Δ is k-fold acylic over some field, then Δ can be decomposed into a disjoint union of rank k boolean intervals whose minimal faces form a subcomplex of Δ.

Stanley's conjecture would provide a combinatorial witness for the complex Δ' found above.

Main Results

Theorem (Duval, Klivans, and Martin, unpublished)

The conjecture is true for $\dim \Delta \leq 2$.

Theorem (Doolittle and Goeckner, 2018)

The conjecture is false in general.

We construct an explicit counterexample with $\dim \Delta = 3$ and $k = 2$. The main theorem relies heavily upon the following theorem, and a particular relative complex.

Theorem (Doolittle and Goeckner, 2018)

Let $\Gamma \subsetneq \Delta$ be complexes such that
- Both Δ and Γ are k-fold acyclic,
- Γ is an induced subcomplex of Δ, and
- the relative complex (Δ, Γ) cannot be decomposed into disjoint rank k boolean intervals.

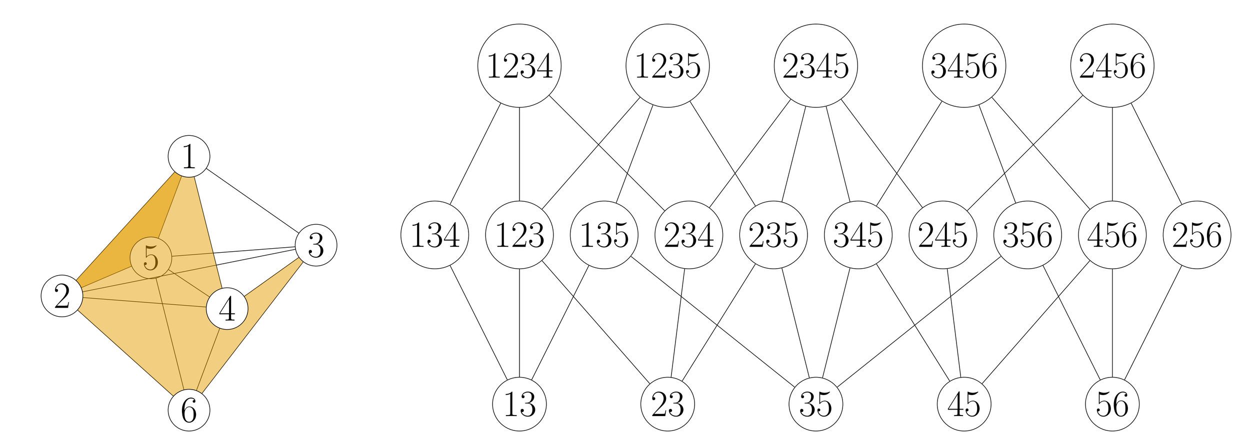
Then gluing many copies of Δ together along Γ produces a k-fold acyclic complex that cannot be decomposed into disjoint rank k boolean intervals.

Remark: Stanley's conjecture holds when $k = \dim \Delta$ (Doolittle and Goeckner, 2018). These complexes are stacked, and all stacked complexes have the appropriate boolean interval decomposition. This generalizes the result of Duval, Klivans, and Martin.

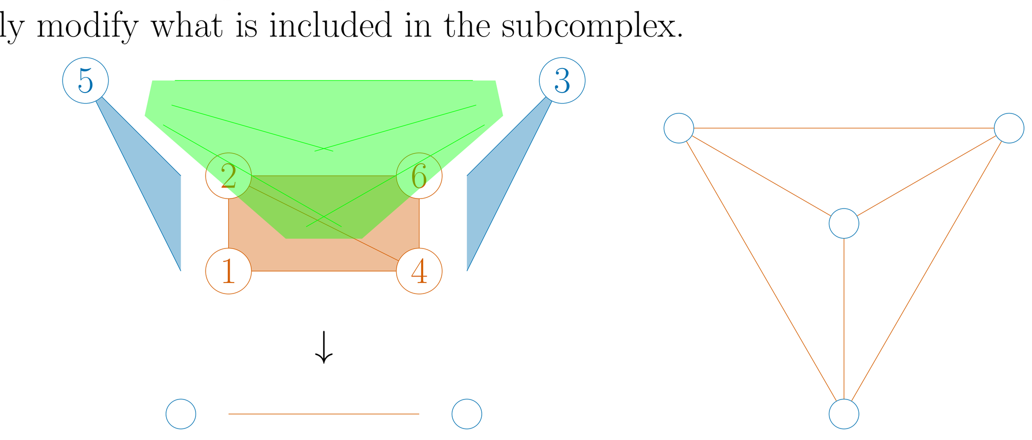
Relative counterexample and the construction

This is the face poset of a *relative* simplicial complex $Z = (X, Y)$ that is *not* decomposable into rank 2 boolean intervals. Both X and Y are 2-fold acyclic.

$$Z = (X, Y) = (\langle 1234, 1235, 2345, 3456, 2456 \rangle, \langle 153, 124, 246, 346 \rangle)$$



This pair satisfies some of the criteria of the theorem, except that Y is not an induced subcomplex of X. To modify this example to something satisfying all the conditions of the theorem, we attach several copies of it together and slightly modify what is included in the subcomplex.



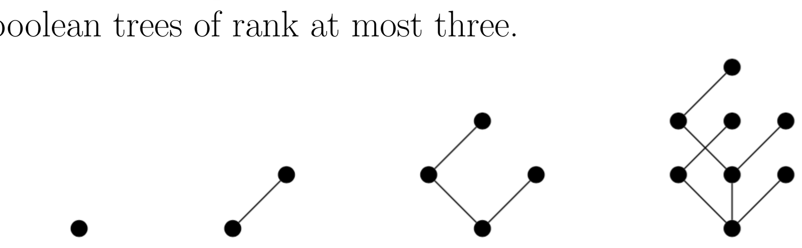
We use all the red-orange faces (along with a few more to preserve 2-fold acyclicity) as Γ and the whole complex as Δ. It is straightforward to verify that (Δ, Γ) satisfies all of the criteria of the main theorem. If $\Omega = \Omega_3$ is the complex obtained by gluing just *three* copies of Δ together along Γ, this is a counterexample to the conjecture. Then

$$f(\Omega, t) = 1 + 20t + 136t^2 + 216t^3 + 99t^4 = (1 + t)^2(1 + 18t + 99t^2)$$

and Ω is 2-fold aylic, but its face poset cannot be decomposed into rank 2 boolean intervals.

Boolean trees

Stanley's conjecture is true if we replace "boolean intervals" with "boolean trees" (Doolittle and Goeckner, 2018). Below are the boolean trees of rank at most three.



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