The rank function and the flats of a positroid

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Abstract

A positroid is a special case of a realizable matroid, that arose from the study of totally nonnegative part of the Grassmannian by Postnikov. Postnikov demonstrated that positroids are in bijection with certain interesting classes of combinatorial objects, such as Grassmann necklaces and decorated permutations. The bases of a positroid can be described directly in terms of the Grassmann necklace and decorated permutation. In this extended abstract, we show how to describe the flats, bases and independent sets directly from the decorated permutation, bypassing the use of the Grassmann necklace.

1. Positroids

- Totally nonnegative (all maximal minors ≥ 0) A:

$$A = \begin{pmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 4 & 0 \end{pmatrix}$$

- Positroid $\mathcal{M}_A = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}$
- Positroid $\leftrightarrow$ decorated permutation (color fixed points with B/W)
- (Ardila-Rincon-Williams) Matroid polytope : $\Sigma_{i\in[a,b]} x_i \leq rk([a,b])$
- Decorated permutation $\leftrightarrow$ Grassmann necklace $\leftrightarrow$ bases
- Our result : Decorated permutation $\leftrightarrow$ rank function (bypass necklaces and bases)

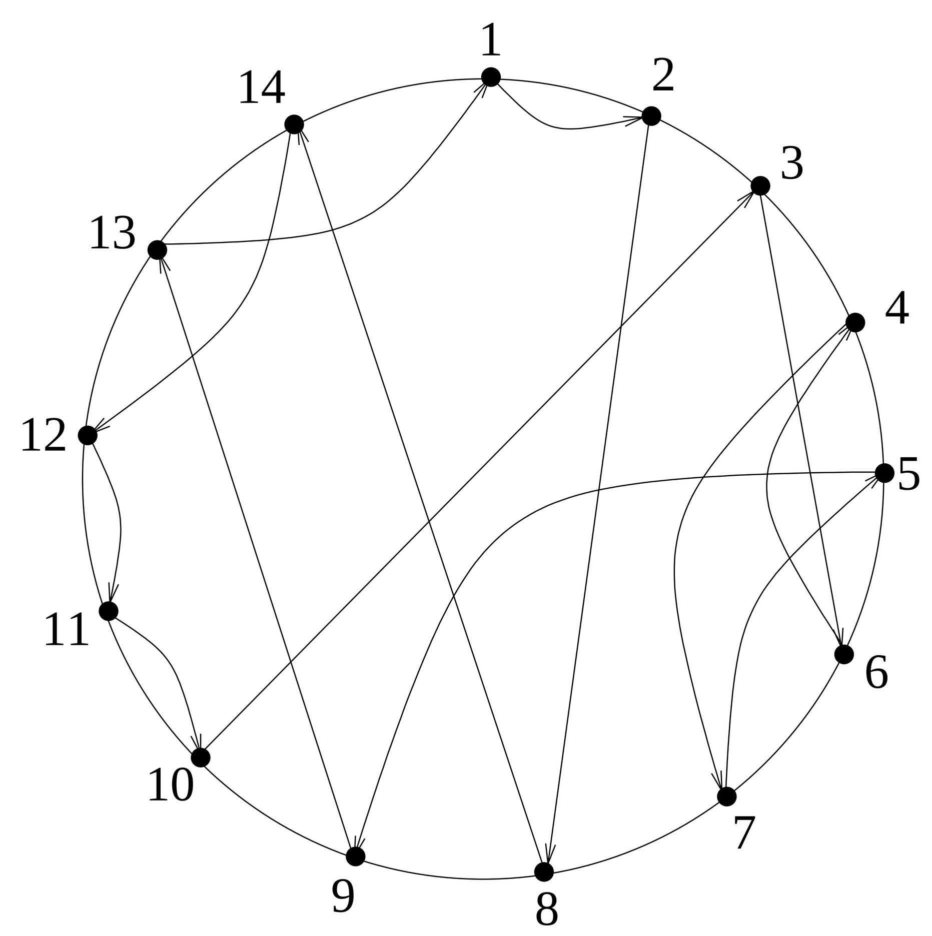
2. Rank function and non-crossing partitions

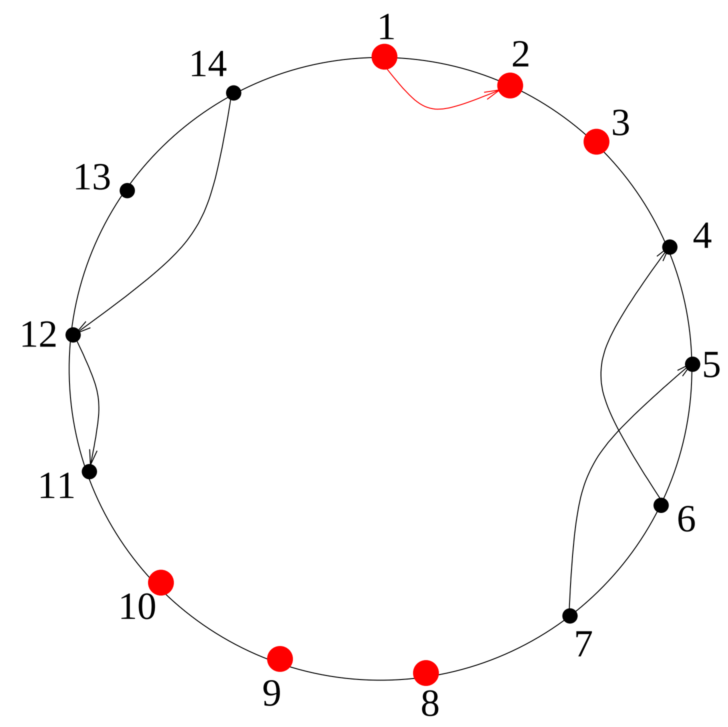
- $[n]$ is our ground set.
- π is the permutation of our positroid.
- CCW-arrow of π : $[x, \pi^{-1}(x)]$
- natural rank bound of $I \subseteq [n]$: $\mathrm{rk}(\mathcal{M})$ minus the number of CCW-arrows contained in I^c.
- $\mathrm{nbd}([a,b]) = |[a,b]| - \mathrm{cw}([a,b])$
- Write $I = [a_1,b_1] \cup \cdots \cup [a_s,b_s]$
- $I_{1,4,5} := [a_1,b_1] \cup [a_4,b_4] \cup [a_5,b_5]$
- $\mathrm{nbd}(I, \{145, 23, 6\}) := \mathrm{nbd}(I_{145}) + \mathrm{nbd}(I_{23}) + \mathrm{nbd}(I_6)$

Rank function

$\mathrm{rk}(I) = \min_{\forall \Pi \in \mathrm{NC}([s])} \mathrm{nbd}(I, \Pi)$, where $\mathrm{NC}([s])$ denotes the set of all non-crossing partitions of $[s]$.

Example : Reading the rank





- $\mathrm{nbd}([1,3] \cup [8,10], \{\{1,2\}\}) = 7-2-2 = 3$
- $\mathrm{nbd}([1,3] \cup [8,10], \{\{1\},\{2\}\}) = \mathrm{nbd}([1,3]) + \mathrm{nbd}([8,10]) = 2+3 = 5$
- Theorem gives $\mathrm{rk}([1,3] \cup [8,10]) = 3$

3. Inseperable flats

- Interval I is a flat iff its I^c coverd by CCW-arrows
- same for Inseparable sets

Matroid polytope

Given by $x_i \geq 0$ for all $i \in [n]$, the equality $x_1 + \cdots + x_n = d$ and inequalities

$$\Sigma_{l\in I} x_l \leq d - \mathrm{ccw}(I^c),$$

where I^c is an interval covered by CCW-arrows.

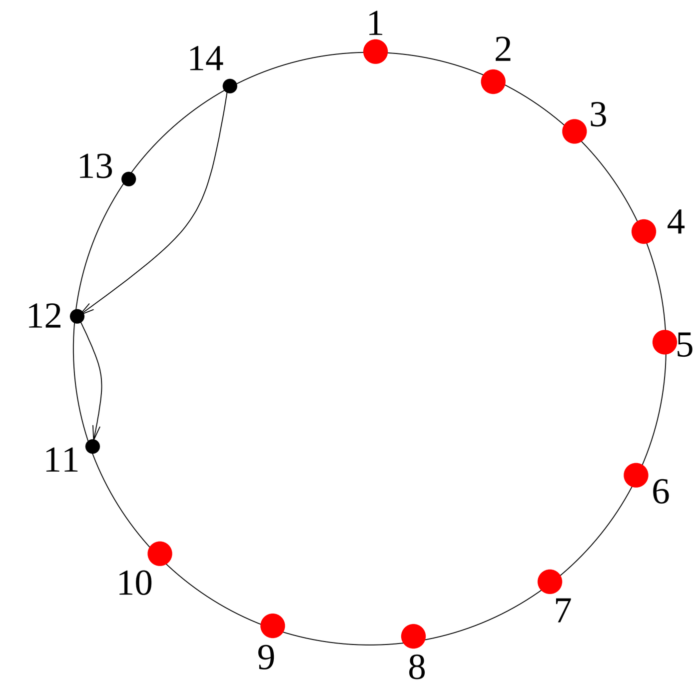
Independent set polytope

Given by $x_i \geq 0$ for all $i \in [n]$ and inequalities

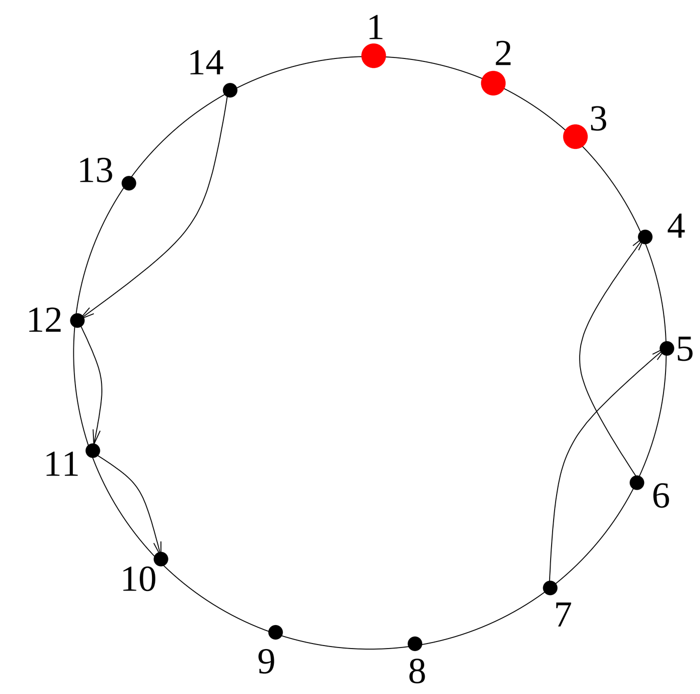
$$\Sigma_{l\in I} x_l \leq d - \mathrm{ccw}(I^c),$$

where I^c is covered by CCW-arrows.

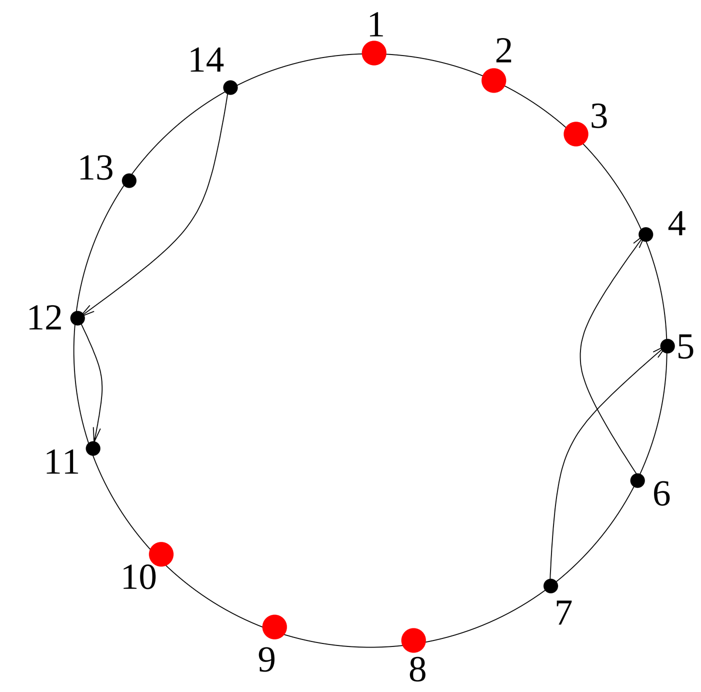
Example : Reading the flats



$[1,10]^c$ is coverd by CCW-arrows : flat!
$x_1 + \cdots + x_{10} \leq 7-2$



$[1,3]^c$ is not covered by CCW-arrows : non-flat!



$[1,3] \cup [8,10]$: flat!
$x_1 + x_2 + x_3 + x_8 + x_9 + x_{10} \leq 7-2-2$

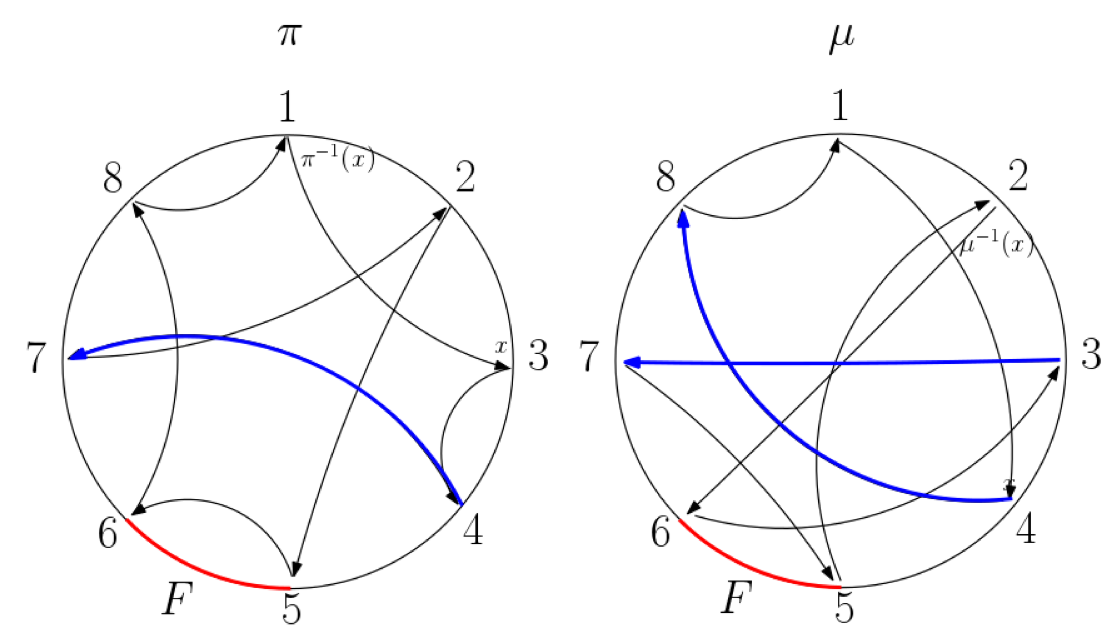
4. Conjectured description of concordancy

- $$A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
- First two rows : positroid $\mathcal{M}_1 = \{12, 13, 14, 23, 34\}$
- All three rows : positroid $\mathcal{M}_2 = \{123, 134\}$
- $\mathcal{M}_1$ and $\mathcal{M}_2$ are concordant! (Can come from same matrix)

Conjecture

Let us have two positroids of rank a and b ($a < b$) each indexed by permutations π and μ. The two positroids can come from the same totally nonnegative matrix by taking the first a rows and first b rows each, if and only if every CCW-arrow of μ is covered by CCW-arrows of π.

Example



Each and every CCW-arrow of μ is covered by CCW-arrows of π! They are concordant!