The rank function and the flats of a positroid
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1. Positroids

- Totally nonnegative (all maximal minors $\geq 0$): $A = \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 4 & 0 \end{bmatrix}$
- Positroid $\mathcal{M}_A = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}
- Positroid $\leftrightarrow$ decorated permutation (color fixed points with B/W)
- (Ardila-Rincon-Williams) Matroid polytope: $\Sigma_{\in \mathcal{M}(A)} \leq rk(a, b)$
- Decorated permutation $\leftrightarrow$ Grassmann necklace $\leftrightarrow$ bases
- Our result: Decorated permutation $\leftrightarrow$ rank function (bypass necklaces and bases)

2. Rank function and non-crossing partitions

- $[n]$ is our ground set.
- $\pi$ is the permutation of our positroid.
- CCW-arrow of $\pi$: $[x, \pi^{-1}(x)]$
- natural rank bound of $I \subseteq [n]$ : $rk(M)$ minus the number of CCW-arrows contained in $I$.
- $nbd([a, b]) = |[a, b]| - ccw([a, b])$ (denotes number of CCW-arrows)
- Write $I = [a_1, b_1] \cup \cdots \cup [a_n, b_n]$
- $I_{\leq 4.5} := [a_1, b_1] \cup [a_2, b_2] \cup [a_3, b_3] \cup [a_4, b_4]$
- $nbd(I, \{145, 23, 6\}) = nbd(I_{145}) + nbd(I_2) + nbd(I_6)$

**Rank function**: $rk(I) = \min_{\Pi \in NC([s])} nbd(I, \Pi)$, where $NC([s])$ denotes the set of all non-crossing partitions of $[s]$.

3. Inseparable flats

- Interval $I$ is a flat iff its $I'$ covered by CCW-arrows
- same for Inseparable sets

**Matroid polytope**

Given by $x_i \geq 0$ for all $i \in [n]$, the equality $x_1 + \cdots + x_n = d$ and inequalities

$\sum_{i \in I} x_i \leq d - ccw(I')$, where $I'$ is an interval covered by CCW-arrows.

**Independent set polytope**

Given by $x_i \geq 0$ for all $i \in [n]$ and inequalities

$\sum_{i \in I} x_i \leq d - ccw(I')$, where $I'$ is covered by CCW-arrows.

4. Conjectured description of concordancy

- $A = \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 4 & 0 \end{bmatrix}$
- First two rows: positroid $\mathcal{M}_1 = \{12, 13, 14, 23, 34\}$
- All three rows: positroid $\mathcal{M}_2 = \{123, 134\}$
- $\mathcal{M}_1$ and $\mathcal{M}_2$ are concordant! (Can come from same matrix)

**Conjecture**

Let us have two positroids of rank $a$ and $b$ ($a < b$) each indexed by permutations $\pi$ and $\mu$.

The two positroids can come from the same totally nonnegative matrix by taking the first $a$ rows and first $b$ rows each, if and only if every CCW-arrow of $\mu$ is covered by CCW-arrows of $\pi$.

Example: Reading the rank

- $nbd([1, 3] \cup [8, 10], \{1, 2\}) = 7 - 2 - 2 = 3$
- $nbd([1, 3] \cup [8, 10], \{1\}, \{2\}) = nbd([1, 3]) + nbd([8, 10]) = 2 + 3 = 5$
- Theorem gives $rk([1, 3] \cup [8, 10]) = 3$

Example: Reading the flats

- $[1, 10]^c$ is covered by CCW-arrows: flat!
  $x_1 + \cdots + x_{10} \leq 7 - 2$
- $[1, 3]^c$ is not covered by CCW-arrows: non-flat!
- $[1, 3]^c$ is covered by CCW-arrows: flat!
  $x_1 + x_2 + x_3 + x_4 + x_9 + x_{10} \leq 7 - 2 - 2$

Example

Each and every CCW-arrow of $\mu$ is covered by CCW-arrows of $\pi$. They are concordant!