Total nonnegativity and Hecke algebra characters

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Total nonnegativity

Call a matrix *totally nonnegative* (TNN) if each of its minors is nonnegative. Call a polynomial $p(x) := p(x_{1,1}, x_{1,2}, \dots, x_{n,n}) \in \mathbb{Z}[x_{1,1}, x_{1,2}, \dots, x_{n,n}]$ *totally nonnegative* (TNN) if $p(A) := p(a_{1,1}, a_{1,2}, \dots, a_{n,n}) \geq 0$ for each TNN matrix $A = (a_{i,j})$.

Question: For what functions $\theta : \mathfrak{S}_n \to \mathbb{Z}$ is the polynomial

$$\mathrm{Imm}_\theta(x) := \sum_{w \in \mathfrak{S}_n} \theta(w) x_{1,w_1} \cdots x_{n,w_n}$$

TNN? Can we combinatorially interpret $\mathrm{Imm}_\theta(A)$?

Fact: For each irreducible character χ^λ, the polynomial $\mathrm{Imm}_{\chi^\lambda}(x)$ is TNN. No combinatorial interpretation for $\mathrm{Imm}_{\chi^\lambda}(A)$ is known.

The Hecke algebra $H_n(q)$

Generated over $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ by $T_{s_1}, \dots, T_{s_{n-1}}$ ($T_e := 1$) with relations

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + qT_e && \text{for } i = 1, \dots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j} && \text{for } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} && \text{for } |i-j| \geq 2. \end{aligned}$$

Natural basis: $\{T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}} \,|\, w = s_{i_1} \cdots s_{i_\ell} \text{ reduced in } S_n\}$.

(Modified) Kazhdan-Lusztig basis: $\{\widetilde{C}_w(q) \,|\, w \in S_n\}$,

$$\widetilde{C}_w(q) = q^{\frac{\ell(w)}{2}} C'_w(q) = \sum_{v \leq w} P_{v,w}(q) T_v.$$

$H_n(1) \cong \mathbb{Z}[\mathfrak{S}_n]$.

Total nonnegativity and linear functionals on $H_n(q)$

Call $s_{[i,j]} = 1 \cdots (i-1) j \cdots i (j+1) \cdots n \in \mathfrak{S}_n$ a *reversal*.

Fact (S '91, H '93): Fix a linear function $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ and its specialization $\theta : \mathfrak{S}_n \to \mathbb{Z}$. Then each of the following statements implies the next:

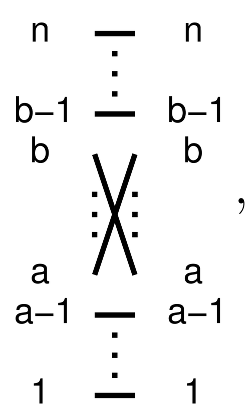
1. $\theta_q(\widetilde{C}_w(q)) \in \mathbb{N}[q]$ for all $w \in \mathfrak{S}_n$.
2. $\theta_q(\widetilde{C}_{s_{[i_1,j_1]}}(q) \cdots \widetilde{C}_{s_{[i_m,j_m]}}(q)) \in \mathbb{N}[q]$ for all reversal sequences $(s_{[i_1,j_1]}, \dots, s_{[i_m,j_m]})$.
3. $\theta(\widetilde{C}_{s_{[i_1,j_1]}}(1) \cdots \widetilde{C}_{s_{[i_m,j_m]}}(1)) \in \mathbb{N}$ for all reversal sequences $(s_{[i_1,j_1]}, \dots, s_{[i_m,j_m]})$.
4. $\mathrm{Imm}_\theta(x)$ is TNN.

Question: For what functions $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ can we combinatorially interpret $\theta_q(\widetilde{C}_{s_{[i_1,j_1]}}(q) \cdots \widetilde{C}_{s_{[i_m,j_m]}}(q))$?

Fact (H '93): For each irreducible character χ_q^λ and each $w \in \mathfrak{S}_n$ we have $\chi_q^\lambda(\widetilde{C}_w(q)) \in \mathbb{N}[q]$. Thus $\chi_q^\lambda(\widetilde{C}_{s_{[i_1,j_1]}}(q) \cdots \widetilde{C}_{s_{[i_m,j_m]}}(q)) \in \mathbb{N}[q]$ for all $\lambda \vdash n$; $w \in \mathfrak{S}_n$. No combinatorial interpretation known.

Star networks and path families

To each Kazhdan-Lusztig basis element $\widetilde{C}_{s_{[a,b]}}(q) \in H_n(q)$ associate the star network

$G_{[a,b]} =$ 

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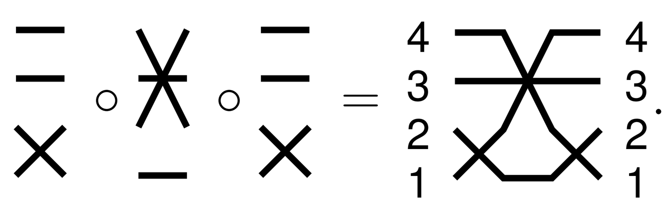
(edges oriented left to right). Associate concatenations to products.

Example: To $\widetilde{C}_{s_{[1,2]}}(q)\widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[1,2]}}(q) \in H_4(q)$, associate the star network

$G_{[1,2]} \circ G_{[2,4]} \circ G_{[1,2]} =$ 

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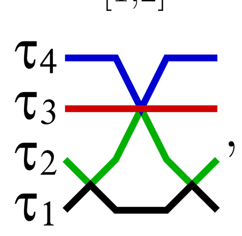
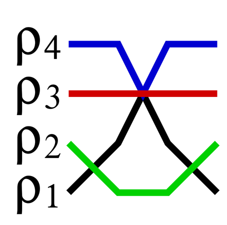
Call $\pi = (\pi_1, \dots, \pi_n)$ a *path family in* G if π_i is a path from source i on left to sink i on right, and π covers all edges of G. Let $\mathrm{c}(\pi) = \#$ crossings in π (always even).

Example: $G = G_{[1,2]} \circ G_{[2,4]} \circ G_{[1,2]}$ is covered by two path families.

$\tau =$ 

, $\mathrm{c}(\tau) = 0$; $\rho =$ 

, $\mathrm{c}(\rho) = 2$.

Path tableaux and inverted noncrossings

A *π-tableau* (*G-tableau*) of shape $\lambda \vdash n$ is an arrangement of $\pi_1, \dots, \pi_n$ into left-justified rows, with λ_i paths in row i. Call a π-tableau *column-strict* if

$$\begin{array}{|c|}\hline \pi_j \\ \hline \pi_i \\ \hline\end{array} \Rightarrow$$ π_i lies entirely below π_j (no shared vertex.)

Example: Let $G = G_{[1,2]} \circ G_{[2,4]} \circ G_{[1,2]}$.
There are 8 column-strict G-tableaux of shape 31.

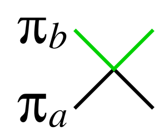
$\begin{array}{lll} \tau_3 & & \\ \tau_1 & \tau_2 & \tau_4 \end{array}$, $U_\tau^{(1)}$; $\begin{array}{lll} \tau_3 & & \\ \tau_1 & \tau_4 & \tau_2 \end{array}$, $U_\tau^{(2)}$; $\begin{array}{lll} \tau_4 & & \\ \tau_1 & \tau_2 & \tau_3 \end{array}$, $U_\tau^{(3)}$; $\begin{array}{lll} \tau_4 & & \\ \tau_1 & \tau_3 & \tau_2 \end{array}$, $U_\tau^{(4)}$

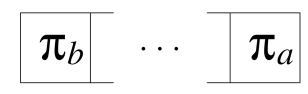
$\begin{array}{lll} \rho_3 & & \\ \rho_2 & \rho_1 & \rho_4 \end{array}$, $U_\rho^{(1)}$; $\begin{array}{lll} \rho_3 & & \\ \rho_2 & \rho_4 & \rho_1 \end{array}$, $U_\rho^{(2)}$; $\begin{array}{lll} \rho_4 & & \\ \rho_2 & \rho_1 & \rho_3 \end{array}$, $U_\rho^{(3)}$; $\begin{array}{lll} \rho_4 & & \\ \rho_2 & \rho_3 & \rho_1 \end{array}$, $U_\rho^{(4)}$

Let π cover star network $G = G_{[i_1,j_1]} \circ \cdots \circ G_{[i_m,j_m]}$ and let U be a π-tableau. If

- π_a, π_b pass through the central vertex of $G_{[i_p,j_p]}$ but do not cross there,
- π_b enters and exits above π_a,
- π_b appears in an earlier column of U than π_a,

then call the triple (p, π_a, π_b) an *inverted noncrossing in* U:



, 

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Let invnc(U) be the number of inverted noncrossings in U.

Example: The 8 tableaux above satisfy the following.

1. invnc$(U_\tau^{(1)}) = 1$: $(2, \tau_3, \tau_2)$.
2. invnc$(U_\tau^{(2)}) = 2$: $(2, \tau_3, \tau_2), (2, \tau_4, \tau_2)$.
3. invnc$(U_\tau^{(3)}) = 2$: $(2, \tau_4, \tau_3), (2, \tau_4, \tau_2)$.
4. invnc$(U_\tau^{(4)}) = 3$: $(2, \tau_4, \tau_3), (2, \tau_4, \tau_2), (2, \tau_3, \tau_2)$.
5. invnc$(U_\rho^{(1)}) = 1$: $(2, \rho_3, \rho_1)$.
6. invnc$(U_\rho^{(2)}) = 2$: $(2, \rho_4, \rho_1), (2, \rho_3, \rho_1)$.
7. invnc$(U_\rho^{(3)}) = 2$: $(2, \rho_4, \rho_1), (2, \rho_4, \rho_3)$.
8. invnc$(U_\rho^{(4)}) = 3$: $(2, \rho_4, \rho_3), (2, \rho_4, \rho_1), (2, \rho_3, \rho_1)$.

Observe that $(1, \tau_2, \tau_1)$ and $(3, \tau_2, \tau_1)$ are noncrossings which are not inverted in tableaux $U_\tau^{(1)}, \dots, U_\tau^{(4)}$.

Main result

Theorem (CS '18): Let $\epsilon_q^\lambda = \mathrm{sgn}\!\uparrow_{H_\lambda(q)}^{H_n(q)}$. Then we have

$$\epsilon_q^\lambda(\widetilde{C}_{s_{[i_1,j_1]}}(q) \cdots \widetilde{C}_{s_{[i_m,j_m]}}(q)) = \sum_\pi q^{\frac{\mathrm{c}(\pi)}{2}} \sum_U q^{\mathrm{invnc}(U)},$$

where the first sum is over path families π in $G_{[i_1,j_1]} \circ \cdots G_{[i_m,j_m]}$, and the second sum is over all column-strict π-tableaux U of shape $\lambda^\top$.

Example: We compute $\epsilon_q^{211}(\widetilde{C}_{s_{[1,2]}}(q)\widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[1,2]}}(q))$, with the 8 tableaux above of shape $211^\top = 31$:

$$\begin{aligned} &q^{\frac{\mathrm{c}(\tau)}{2}}(q^{\mathrm{invnc}(U_\tau^{(1)})} + q^{\mathrm{invnc}(U_\tau^{(2)})} + q^{\mathrm{invnc}(U_\tau^{(3)})} + q^{\mathrm{invnc}(U_\tau^{(4)})}) \\ &+ q^{\frac{\mathrm{c}(\rho)}{2}}(q^{\mathrm{invnc}(U_\rho^{(1)})} + q^{\mathrm{invnc}(U_\rho^{(2)})} + q^{\mathrm{invnc}(U_\rho^{(3)})} + q^{\mathrm{invnc}(U_\rho^{(4)})}) \\ &= q^{\frac{0}{2}}(q^1 + 2q^2 + q^3) + q^{\frac{2}{2}}(q^1 + 2q^2 + q^3) \\ &= q + 3q^2 + 3q^3 + q^4. \end{aligned}$$

Open question

Question: Is there a similar combinatorial interpretation of

$$\chi_q^\lambda(\widetilde{C}_{s_{[i_1,j_1]}}(q) \cdots \widetilde{C}_{s_{[i_m,j_m]}}(q)),$$

and therefore of $\mathrm{Imm}_{\chi^\lambda}(A)$ for all TNN matrices A?

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