Connecting descent and peak polynomials

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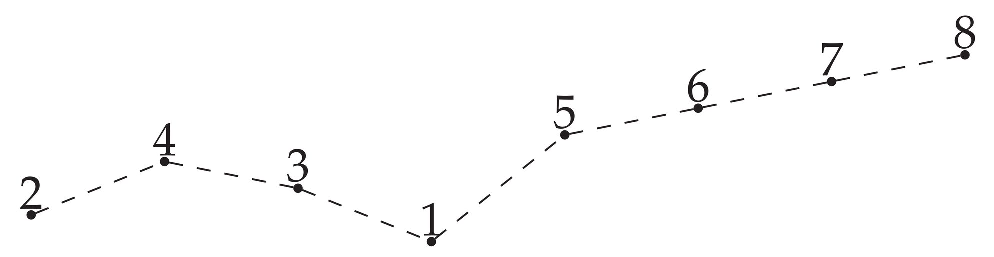
Abstract

The size of the set of all permutations of n with a given descent set is a polynomial in n, called the descent polynomial. Similarly, the size of the set of all permutations of n with a given peak set, adjusted by a power of 2 gives a polynomial in n, called the peak polynomial. In this work we give a unitary expansion of descent polynomials in terms of peak polynomials. Then we use this expansion to give a combinatorial interpretation of the coefficients of the peak polynomial in a binomial basis, thus giving a new proof of the peak polynomial positivity conjecture.

Descents and Peaks

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of $[n]$ written in one-line notation.



Example: $\sigma = 2435678$

We define the descent, peak, valley and spike sets of σ as follows:

- $\mathrm{Des}(\sigma) = \{i \mid \sigma_i > \sigma_{i+1}\}$,
- $\mathrm{Peak}(\sigma) = \{\sigma \mid \sigma_i > \sigma_{i+1}, \sigma_{i-1}\}$,
- $\mathrm{Valley}(\sigma) = \{\sigma \mid \sigma_i < \sigma_{i+1}, \sigma_{i-1}\}$,
- $\mathrm{Spike}(\sigma) = \mathrm{Peak}(\sigma) \cup \mathrm{Valley}(\sigma)$.

$\sigma = 2435678$ has:
$\mathrm{Des}(\sigma) = \{2, 3\}$,
$\mathrm{Peak}(\sigma) = \{2\}$,
$\mathrm{Spike}(\sigma) = \{2, 4\}$.

We set:

- $D(S, n) = \{$ permutations of n with descent set $S\}$.
- $d(S, n) = |D(S, n)|$.
- $P(I, n) = \{$ permutations of n with peak set $I\}$.
- $p(I, n) = |P(I, n)|2^{|I|+1-n}$.

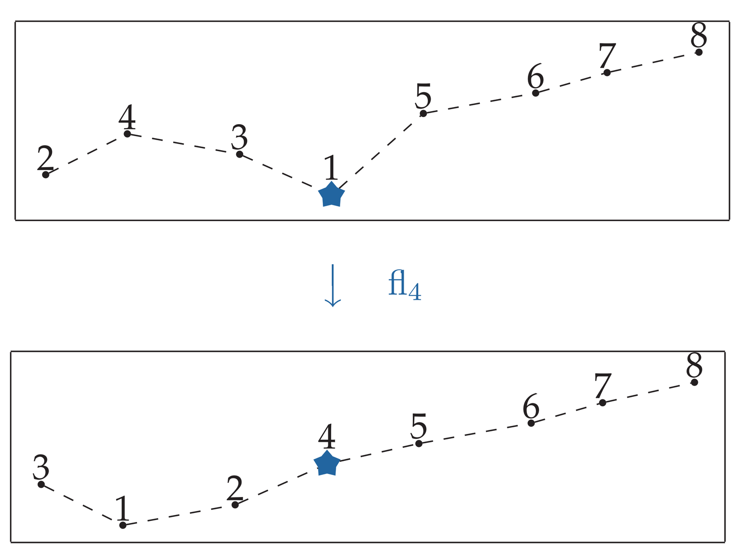
Note that $P(I, n) = \varnothing$ unless I is an admissible set: $I \subset [n-1]/\{1\}$ and I does not contain consecutive elements.
Both $d(S, n)$ and $p(I, n)$ are polynomials in n, called the descent and peak polynomials respectively.

$$
\begin{aligned}
d(\varnothing, n) &= 1 & p(\varnothing, n) &= 1\\
d(\{1\}, n) &= n-1 & p(\{2\}, n) &= n-2\\
d(\{k\}, n) &= \binom{n}{k} - 1 & p(\{k\}, n) &= \binom{n-1}{k-1} - 1
\end{aligned}
$$

Definition: i-flip

Let $\sigma \in S_n$. Let $i \le n$, and

$\{\sigma_1, \sigma_2, \ldots, \sigma_i\}$
$= \{a_1 < \cdots < a_i\}$.



We define fl_i as follows:

$$\mathrm{fl}_i(\sigma)_j = \begin{cases} a_{i-k+1} & \sigma_j = a_k \\ \sigma_j & j > i. \end{cases}$$

For $i \in \mathrm{Spike}(\sigma)$, σ is admits an
i^+-flip: if $\mathrm{Spike}(\mathrm{fl}_i(\sigma)) = \mathrm{Spike}(\sigma)/\{i\}$,
i^--flip: if $\mathrm{Spike}(\mathrm{fl}_{i-1}(\sigma)) = \mathrm{Spike}(\sigma)/\{i\}$.

We say σ admits an i-flip if it admits an i^+- or i^-- flip.

Main Result I

Theorem: Let σ be a permutation with descent set S. We have:

$$d(S, n) = \sum_{I \subset \mathrm{Spike}(\sigma)} p(I, n).$$

Corollary: For any admissible set I,

$$p(I, n) = \sum_{J \subset I} (-1)^{|I|-|J|} d(S_J, n).$$

Definition: S_I

For any set $I \in [n-1]/\{1\}$, let σ be a permutation of n satisfying $\mathrm{Spike}(\sigma) = I$, constructed by alternating the elements of I to be peaks and valleys such that the rightmost one is not a peak. We will use the notation S_I to denote the descent set of σ. S_I only depends on I.
Example: For $I = \{2, 4\}$, $S_I = \{2, 3\}$ is the unique descent set that has 2 as a peak and 4 as a valley.

Theorem [3]

For any finite set of positive integers S with $\max(S) \le m$ we have:

$$d(S, n) = a_0(S)\binom{n-m}{0} + a_1(S)\binom{n-m}{1} + \cdots + a_m(S)\binom{n-m}{m},$$

where the constant $a_k(S)$ is the number of $\sigma \in D(S, 2m)$ such that:

$$\{\sigma_1, \sigma_2, \ldots, \sigma_m\} \cap [m+1, 2m] = [m+1, m+k].$$

Main Result II

Theorem: For any admissible set of I with $\max(I) \le m$ we have

$$p(I, n) = b_0(I)\binom{n-m}{0} + b_1(I)\binom{n-m}{1} + \cdots + b_m(I)\binom{n-m}{m},$$

where the constant $b_k(I)$ is the number of $\sigma \in D(S_I, 2m)$ such that:

$$\{\sigma_1, \sigma_2, \ldots, \sigma_m\} \cap [m+1, 2m] = [m+1, m+k],$$

and σ does not admit any i-flips.

Example

	2-fl	4-fl
14325678	✗	✓
24315678	✗	✓
34215678	✓	✓
57612348	✗	✗
675123478	✓	✗

$k = 0, 3$

	2-fl	4-fl
15324678	✗	✓
15423678	✗	✗
25314678	✗	✗
25413678	✗	✗
35214678	✓	✗
35412678	✗	✗
45213678	✓	✗
45312678	✓	✗

$k = 1$

	2-fl	4-fl
16523478	✗	✗
26513478	✗	✗
36512478	✗	✗
46512378	✗	✗
56213478	✓	✗
56312478	✓	✗
56412378	✓	✗

$k = 2$

$$p(\{2,4\}, n) = 0\binom{n-4}{0} + 4\binom{n-4}{1} + 4\binom{n-4}{2} + 1\binom{n-4}{3}.$$

References

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