Kashiwara’s crystal graphs (cont.)

Fact. \( V(\lambda) \) has a crystal basis \( B(\lambda) \); when \( q \to 0 \) we have
\[
 f_i \cdot e_j = B(\lambda) 
\]
Encode as colored directed graph
\[
f_i(b) = b' \iff b \rightarrow b'.
\]
Example. \( \mathfrak{g} = \mathfrak{sl}_2, \lambda = (3, 3, 1), \text{blue: } e_1, e_2, e_3, \text{ red: } f_1, f_2, f_3\).

The combinatorial atomic decomposition

Let \( B(\lambda)^n \subset B(\lambda) \) consist of dominant weight vertices.

**Definition.** An atomic decomposition of \( B(\lambda) \) is a partition
\[
 B(\lambda)^n = \bigcup_{b \in B(\lambda)} B(\lambda, b),
\]
where \( H(\lambda) \subset B(\lambda)^n \), \( b \in B(\lambda) \) is a distinguished vertex, and \( B(\lambda, b) \) contains exactly one vertex of dominant weight \( \nu \), for \( \nu \leq wt(b) \).

In particular, if \( wt(b) = \mu \), then
\[
w^\nu_b = \sum_{\nu \in B(\lambda, b)} x^\nu.
\]

**Definition.** A \( t \)-atomic decomposition of \( B(\lambda) \) is also endowed with a statistic \( c : H(\lambda) \to \mathbb{Z}_{\geq 0} \) satisfying
\[
A_{\lambda, b}(\mu) = \sum_{\nu \in B(\lambda, b)} c^\nu.
\]

Main ingredients for the atomic decomposition

- **Various properties of the dominance order** — studied by Stembridge [98], we derive additional structural properties in classical types;
- A modified crystal graph structure on the vertices of \( B(\lambda)^n \) and its properties.

Modified crystal structure

Assume that the Dynkin diagram consists of a type \( A_{n-1} \) part, labeled \( 1, \ldots, r - 1 \), and an extra node labeled \( r \).

**Definition.** Given any positive root \( \alpha \in W_0 \), consider \( w \in W \) satisfying \( \epsilon_0(\alpha) = \alpha \) of smallest length, and let
\[
\ell^w_{\alpha} = \ell^w_0 - \ell^w_{\alpha}.
\]

**Definition.** Endow \( B(\lambda)^n \) with a modified crystal graph structure, by restricting to those arrows
\[
b \rightarrow \ell^w_{\alpha}(b) \quad \text{for which } (w(\alpha), b) \in B(\lambda)^n \quad \text{is a co-core}.
\]
We studied relations between \( \ell^w_{\alpha}(b) \).

**Theorem.** (LeCouvey, L.) Under certain conditions
\[
\ell^w_{\alpha}(b) = \ell^w_{\alpha}(b)^{w_0} = 0 \quad \text{if } (\alpha, b) \in W(\alpha, \alpha_2),
\]
\[
\ell^w_{\alpha}(b) = \ell^w_{\alpha}(b)^{w_0} = 0 \quad \text{if } (\alpha, b) \in W(\alpha, \alpha_1).
\]

Main result in types \( A_{n-1}, \, C_n, \, D_n \)

Fix a partition \( \lambda \) — dominant weight. In type \( C_n \), assume
\[ n > (|\lambda| + 1)/2 \]
and \( n > |\lambda| \) in type \( D_n \) (stable range).

**Theorem.** (LeCouvey, L.) The connected components of \( B(\lambda)^n \) are isomorphic to intervals \([\bar{0}, \bar{\mu}]\) in the dominance order, via the projection sending vertices to their weights.

This is a \( t \)-atomic decomposition of \( B(\lambda) \) in type \( A_{n-1} \), and an atomic decomposition in types \( C_n \) and \( D_n \).

Idea of proof

- Consider the “small intervals” of the dominance order (rhombi, pentagons, or hexagons).
- Verify the commutation of the modified crystal operators on these intervals.
- Use this property to iteratively lift the structure of the dominance order to that of the modified crystal poset.

Type \( B_n \)

**Complication.** Even in the stable case, we have covers labeled by a root \( \epsilon_1, \ldots, (10 \times 10)^t \) (or \( b^t \)).

Since \( \epsilon_1 \in B(1) \) (short root), we need \( \ell^w_{\alpha} \) for \( \alpha = \epsilon_1 \).

**Solution.** For \( w \in W \) of smallest length with \( w(\alpha_0) = \alpha = \epsilon_1 \), let
\[
\ell^w_{\alpha} = w f_{\alpha \epsilon_1}^{-1}.
\]

**Theorem.** (LeCouvey, L.) Simulare to types \( A_{n-1}, \, C_n, \, D_n \), in the corresponding stable range.

Example

\[
B(\lambda)^n \quad \text{for } \lambda = (3, 2, 1) \text{ in type } A_2 \text{ as SSYT:}
\]

\[
\begin{array}{c}
(1, 2) \\
(1, 3) \\
(2, 3) \\
(1, 2, 3)
\end{array}
\]

We get the atomic decomposition of the character:
\[
\chi^\lambda = w^\lambda_{(1,2),1} + w^\lambda_{(2,3),2} + w^\lambda_{(3,1,1),3} + w^\lambda_{(2,2,1),4}.
\]

Geometric interpretation: the geometric Satake correspondence

For a reductive group \( G \), it realizes geometrically \( V(\lambda) \) for \( G'_2 \), as the intersection cohomology \( H^\bullet(\mathcal{S}(\hat{G})) \) of a Schubert variety in the affine Grassmannian.

\( H^\bullet(\mathcal{S}(\hat{G})) \) has the truncation filtration:
\[
H^\bullet(\mathcal{S}(\hat{G})) \approx H^\bullet(\mathcal{S}(\hat{G})) \subset \text{other summands}.
\]
This gives \( K_{\lambda, b}(b) \) when restricted to the weight spaces.

\( H^\bullet(\mathcal{S}(\hat{G})) \) has a basis of classes of Schubert varieties inside \( \mathcal{X}(\hat{G}) \), which are indexed by \( b \in P(\lambda) \).

**Interpretation.** According to the atomic decomposition \( \chi^\lambda = \sum A_{\lambda, b}(\mu) w^\mu \), where \( w^n = \sum x^n \), there is a refinement of the truncation filtration, with successive quasipositive isomorphisms to \( H^\bullet(\mathcal{S}(\hat{G})) \), \( b \in P(\lambda) \).

Future work

- Extend the results to affine type \( A \) (with A. Schulze).
- Defining on \( B(\lambda)^n \) a statistic computing \( K_{\lambda, b}(b) \), constructed recursively on the components, starting from its value on the minimal vertex (calculated in [LIL18a]).

References


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