Atomic Decomposition of Characters and Crystals

Cédric Lecouvey† (Univ. of Tours, France), Cristian Lenart† (State Univ. of New York at Albany, USA)

Lie algebras and their representations

Consider a complex semisimple Lie algebra $\mathfrak{g}$.

$R = R^+ \sqcup R^-$ root system, P weight lattice, P^+ dominant weights, W Weyl group.

For $\lambda \in P^+$, let $V(\lambda)$ be the irreducible representation with highest weight λ, and $P(\lambda)$ its weights.

For $\mu \in P(\lambda)$, let $K_{\lambda,\mu}$ be the multiplicity of μ in $V(\lambda)$; in type A it counts SSYT of shape λ and content μ.

Lusztig defined the t-analogue $K_{\lambda,\mu}(t)$, i.e., $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$, via

$$\frac{\sum_{w\in W} \mathrm{sgn}(w)\, x^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R^+}(1-tx^{-\alpha})} = \sum_{\mu\in P(\lambda)} K_{\lambda,\mu}(t)\, x^{\mu}\,.$$

$K_{\lambda,\mu}(t)$, for λ, $\mu \in P^+$, is a Kostka-Foulkes polynomial.

This polynomial has remarkable properties. In particular, it is a special affine Kazhdan-Lusztig polynomial, which implies that it is in $\mathbb{Z}_{\geq 0}[t]$.

We will study a less understood property: the atomic decomposition (only defined in type A by A. Lascoux [L91]).

Basic definitions

The dominance order $\leq$ on P^+ is defined by:

$$\mu \leq \lambda \ \text{ if } \ \lambda - \mu \ \text{ is a } \mathbb{Z}_{\geq 0}\text{-combination of simple roots.}$$

Set

$$P^+(\lambda) := P(\lambda) \cap P^+ = \{\mu \in P^+ \mid \mu \leq \lambda\}\,.$$

Layer sum polynomials:

$$w_\mu^+ := \sum_{\nu \in P^+(\mu)} x^\nu = \sum_{\nu \leq \mu} x^\nu\,.$$

Let

$$\widetilde{K}_{\lambda,\mu}(t) := t^{\langle \lambda-\mu, \rho^\vee\rangle} K_{\lambda,\mu}(t^{-1})\,.$$

The dominant part of the t-character:

$$\chi_\lambda^+(t) := \sum_{\mu \in P^+(\lambda)} \widetilde{K}_{\lambda,\mu}(t)\, x^\mu\,.$$

The atomic decomposition

Consider the atomic polynomials $A_{\lambda,\mu}(t) \in \mathbb{Z}[t]$, defined by one of the following equivalent relations:

$$\chi_\lambda^+(t) = \sum_{\mu\in P^+(\lambda)} A_{\lambda,\mu}(t)\, w_\mu^+\,;$$

$$\widetilde{K}_{\lambda,\nu}(t) = \sum_{\nu\leq\mu\leq\lambda} A_{\lambda,\mu}(t) \quad \text{for all } \nu \leq \lambda\,.$$

Definition. The t-character $\chi_\lambda^+(t)$ (or, equivalently, the Kostka-Foulkes polynomials $K_{\lambda,\nu}(t)$) have a t-atomic decomposition if $A_{\lambda,\mu}(t) \in \mathbb{Z}_{\geq 0}[t]$.

The irreducible character χ_λ has an atomic decomposition if $A_{\lambda,\mu}(1) \in \mathbb{Z}_{\geq 0}$.

Remarks. Goal

(1) Not all irreducible characters have atomic decompositions, but the failures seem limited to small ranks.

(2) All type A t-characters (Kostka-Foulkes polynomials) have t-atomic decompositions – Lascoux [L91], proof by Shimozono based on intricate tableau combinatorics: *plactic monoid, cyclage, catabolism.*

(3) The t-atomic decomposition of Kostka-Foulkes polynomials is a strengthening of their monotonicity:

$$\widetilde{K}_{\lambda,\nu}(t) - \widetilde{K}_{\lambda,\mu}(t) \in \mathbb{Z}_{\geq 0}[t]\,, \qquad \text{for } \nu \leq \mu \leq \lambda\,.$$

Goal. Simpler, more conceptual approach to the atomic decomposition, which extends beyond type A.

Solution. Define a combinatorial decomposition, based on Kashiwara's crystal graphs.

Kashiwara's crystal graphs

Encode irreducible representations $V(\lambda)$ of the corresponding quantum group $U_q(\mathfrak{g})$ as $q \to 0$.

Kashiwara (crystal) operators are modified versions of the Chevalley generators: e_i, f_i, $i \in I$.

Kashiwara's crystal graphs (cont.)

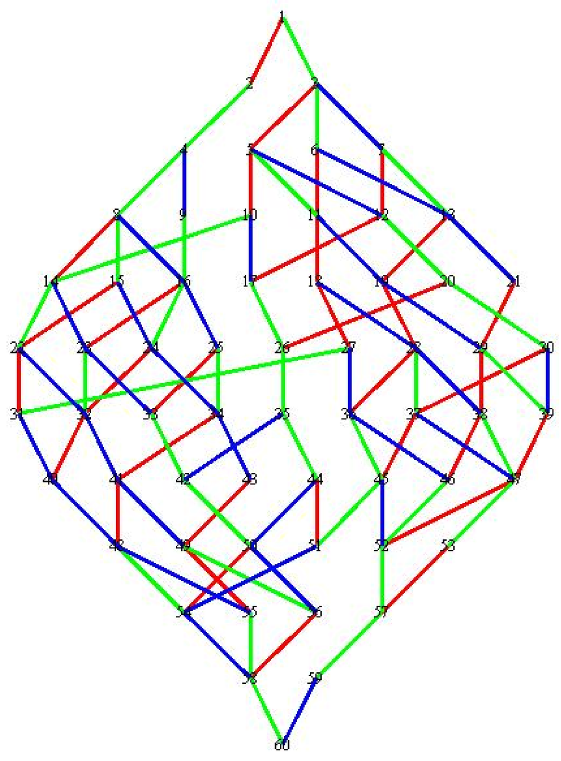
Fact. $V(\lambda)$ has a crystal basis $B(\lambda)$: when $q \to 0$ we have

$$f_i, e_i \ : \ B(\lambda) \to B(\lambda) \sqcup \{\mathbf{0}\}\,.$$

Encode as colored directed graph:

$$f_i(b) = b' \quad \Longleftrightarrow \quad b \stackrel{i}{\longrightarrow} b'\,.$$

Example. $\mathfrak{g} = \mathfrak{sl}_4$, $\lambda = (3,3,1)$, blue: $\alpha_1 = \varepsilon_1 - \varepsilon_2$, green: $\alpha_2 = \varepsilon_2 - \varepsilon_3$, red: $\alpha_3 = \varepsilon_3 - \varepsilon_4$.



The combinatorial atomic decomposition

Let $B(\lambda)^+ \subset B(\lambda)$ consist of dominant weight vertices.

Definition. An atomic decomposition of $B(\lambda)$ is a partition

$$B(\lambda)^+ = \bigsqcup_{h\in H(\lambda)} B(\lambda, h)\,,$$

where $H(\lambda) \subset B(\lambda)^+$, $h \in B(\lambda,h)$ is a distinguished vertex, and $B(\lambda,h)$ contains exactly one vertex of dominant weight ν, for $\nu \leq \mathrm{wt}(h)$.

In particular, if $\mathrm{wt}(h) = \mu$, then

$$w_\mu^+ = \sum_{b\in B(\lambda,h)} x^{\mathrm{wt}(b)}\,.$$

Definition. A t-atomic decomposition of $B(\lambda)$ is also endowed with a statistic $\mathrm{c} : H(\lambda) \to \mathbb{Z}_{\geq 0}$ satisfying

$$A_{\lambda,\mu}(t) = \sum_{\substack{h\in H(\lambda)\\ \mathrm{wt}(h)=\mu}} t^{\mathrm{c}(h)}\,.$$

Main ingredients for the atomic decomposition

- various properties of the dominance order – studied by Stembridge [S98], we derive additional structural properties in classical types;
- a modified crystal graph structure on the vertices of $B(\lambda)^+$ and its properties.

Modified crystal structure

Assume that the Dynkin diagram consists of a type A_{r-1} part, labeled $1, \ldots, r-1$, and an extra node labeled r.

Definition. Given *any* positive root $\alpha \in W\alpha_1$, consider $w \in W$ satisfying $w(\alpha_1) = \alpha$ of smallest length, and let

$$\widehat{f}_\alpha := w f_1 w^{-1}\,.$$

Definition. Endow $B(\lambda)^+$ with a modified crystal graph structure, by restricting to those arrows

$$b \to \widehat{f}_\alpha(b) \quad \text{for which} \quad \mathrm{wt}(b) \gtrdot \mathrm{wt}(\widehat{f}_\alpha(b)) \quad \text{is a cocover}\,.$$

We studied relations between $\widehat{f}_\alpha$ on $B(\lambda)^+$.

Theorem. (Lecouvey, L.) *Under certain conditions:*

$$\widehat{f}_\alpha \widehat{f}_\beta(b) = \begin{cases} \widehat{f}_\beta \widehat{f}_\alpha(b) = \widehat{f}_{\alpha+\beta}(b) \neq \mathbf{0} & \text{if } (\alpha,\beta) \in W(\alpha_1,\alpha_2)\,, \\ \widehat{f}_\beta \widehat{f}_\alpha(b) \neq \mathbf{0} & \text{if } (\alpha,\beta) \in W(\alpha_1,\alpha_3)\,. \end{cases}$$

Main result in types A_{n-1}, C_n, D_n

Fix a partition λ – dominant weight. In type C_n assume $n > (|\lambda|+1)/2$, and $n > |\lambda|$ in type D_n (stable range).

Theorem. (Lecouvey, L.) *The connected components of $\widehat{B}(\lambda)^+$ are isomorphic to intervals $[\widehat{0}, \mu]$ in the dominance order, via the projection sending vertices to their weights.*

This is a t-atomic decomposition of $B(\lambda)$ in type A_{n-1}, and an atomic decomposition in types C_n and D_n.

Idea of proof

- Consider the "small intervals" of the dominance order (rhombi, pentagons, or hexagons).
- Verify the commutation of the modified crystal operators on these intervals.
- Use this property to iteratively lift the structure of the dominance order to that of the modified crystal poset.

Type B_n

Complication. Even in the stable case, we have covers labeled by a root ε_i: $(\ldots 10^k) \gtrdot (\ldots 0^{k+1})$.

Since $\varepsilon_i \notin W\alpha_1$ (short root), we also need $\widehat{f}_\alpha$ for $\alpha = \varepsilon_i$.

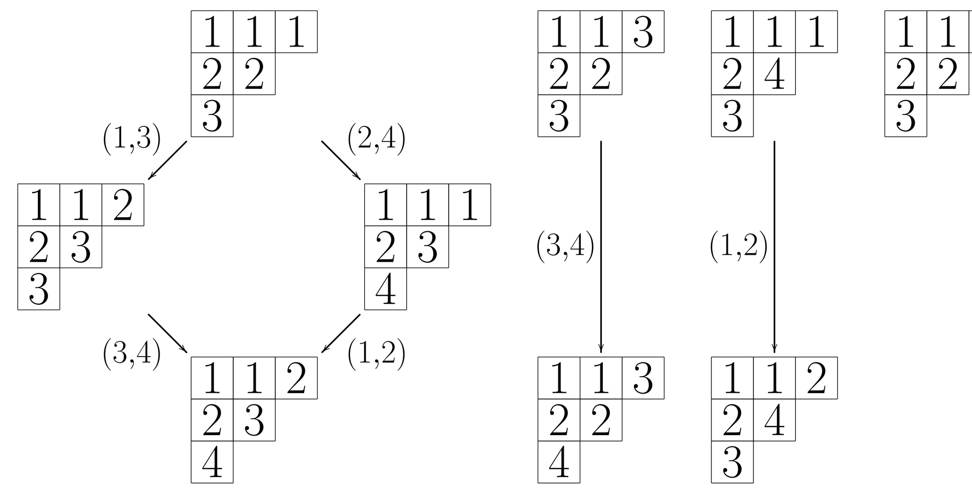
Solution. For $w \in W$ of smallest length with $w(\alpha_n) = \alpha = \varepsilon_i$, let

$$\widehat{f}_\alpha := w f_n w^{-1}\,.$$

Theorem. (Lecouvey, L.) *Similar to types A_{n-1}, C_n, D_n, in the corresponding stable range.*

Example

$B(\lambda)^+$ for $\lambda = (3,2,1)$ in type A_3, as SSYT:



We get the atomic decomposition of the character:

$$\chi_\lambda^+ = w^+_{(3,2,1)} + w^+_{(2,2,2)} + w^+_{(3,1,1,1)} + w^+_{(2,2,1,1)}\,.$$

Geometric interpretation: the geometric Satake correspondence

For a reductive group G, it realizes geometrically $V(\lambda)$ for $G^\vee$, as the intersection cohomology $IH^*(\overline{Gr_\lambda})$ of a Schubert variety in the affine Grassmannian.

$IH^*(\overline{Gr^\lambda})$ has the truncation filtration:

$$IH^*(\overline{Gr^\lambda}) \simeq H^*(\overline{Gr^\lambda}) \oplus \text{other summands}\,.$$

This gives $K_{\lambda,\mu}(t)$ when restricted to the weight spaces.

$H^*(\overline{Gr^\lambda})$ has a basis of classes of Schubert varieties inside $\overline{Gr_\lambda}$, which are indexed by $\mu \in P(\lambda)$.

Interpretation. *According to the atomic decomposition*

$$\chi_\lambda^+(t) = \sum_{\mu\in P^+(\lambda)} A_{\lambda,\mu}(t)\, w_\mu^+\,, \quad \text{where} \quad w_\mu^+ := \sum_{\nu\in P^+(\mu)} x^\nu\,,$$

there is a refinement of the truncation filtration, with successive quotients isomorphic to $H^*(\overline{Gr^\mu}), \mu \in P^+(\lambda)$.

Future work

- Extend the results to affine type A (with A. Schultze).
- Defining on $B(\lambda^+)$ a statistic computing $K_{\lambda,\mu}(t)$: constructed recursively on the components, starting from its value on the minimal vertex (calculated in [LL18a]).

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Contact: cedric.lecouvey@lmpt.univ-tours.fr, clenart@albany.edu

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