Topological Bijections for Oriented Matroids

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Introduction

- We introduce a family of bijections between bases and special orientations of an oriented matroid, based on the picture of *Topological Representation Theorem*.
- This generalizes the core construction in Backman–Baker–Yuen (FPSAC 2018) on bijections between bases and the *Jacobian group* of a regular matroid.
- Connections with *Bernardi bijections* of planar maps, and *orientation activity* of Gioan and Las Vergnas.

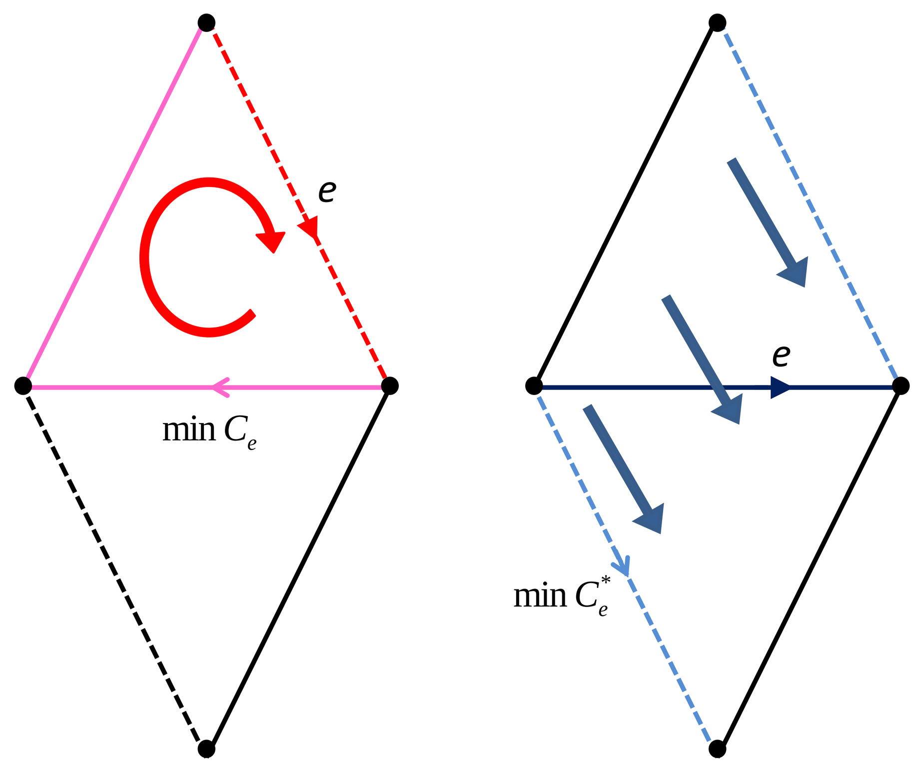
Setting

- M: an oriented matroid on E.
- $M' = M \sqcup \{f\}, \widetilde{M} = M \sqcup \{g\}$: *generic* single-element extension (resp. lift) of M.
- For each circuit (resp. cocircuit) C of M, $\sigma(C)$ (resp. $\sigma^*(C)$) is the unique orientation of C such that the sign of g (resp. f) in its lift (resp. extension) is $+$.
- An orientation $\mathcal{O}$ of M is (σ,σ^*)*-compatible* if every signed circuit (resp. cocircuit) compatible with $\mathcal{O}$ is oriented according to σ (resp. σ^*).

Important Example (*lexicographic data*): Fix a total ordering $<$ of E together with a reference orientation of E. Orient each circuit (resp. cocircuit) according to the reference orientation of its minimal element.

Theorem (BSY 2018+)

Given a basis B, let $\mathcal{O}(B)$ be the orientation of M in which we orient each $e \notin B$ according to its orientation in $\sigma(C(B,e))$ and each $e \in B$ according to its orientation in $\sigma^*(C^*(B,e))$. Then $\beta_{\sigma,\sigma^*}: B \mapsto \mathcal{O}(B)$ is a bijection between bases and (σ,σ^*)-compatible orientations of M.



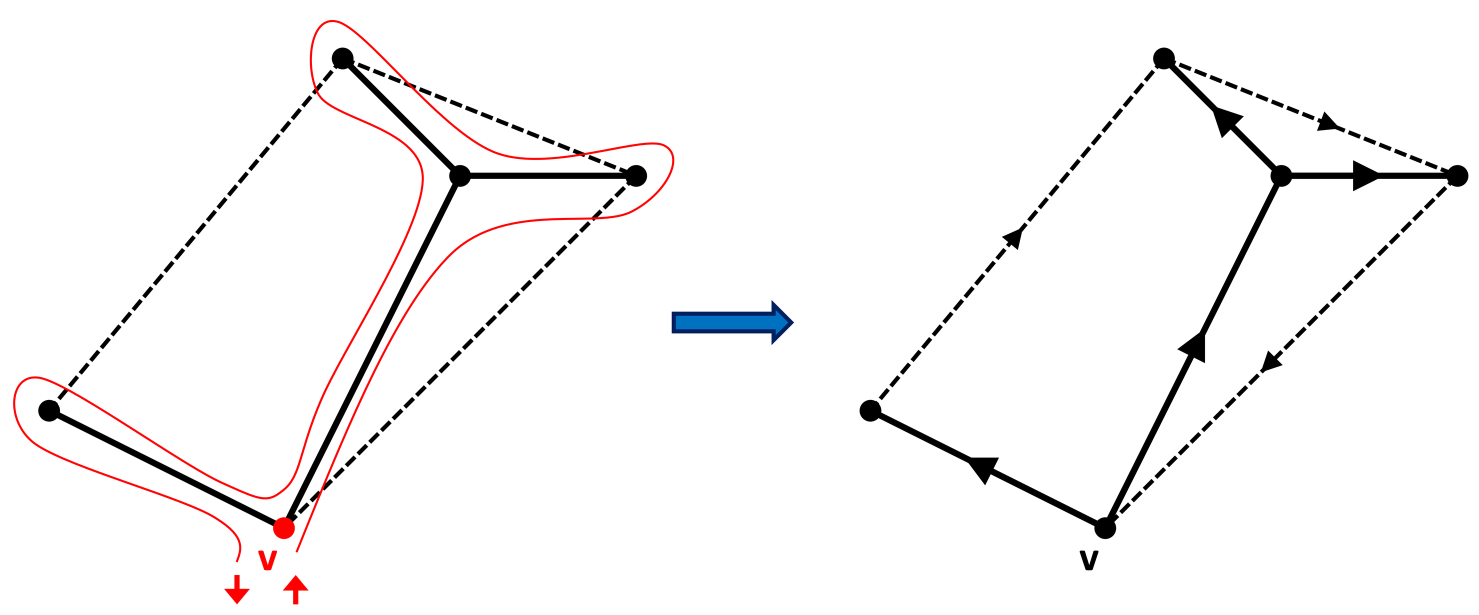
Corollary

The number of (σ,σ^*)-compatible orientations of an oriented matroid equals the number of bases for any pair of (generic) σ,σ^*.

Remark: Compare with the theorem of Greene–Zaslavsky on the number of bounded regions in a hyperplane arrangement.

Planar Bernardi Bijections

Let G be a planar map with a root v. Given a spanning tree T of G, walk counter-clockwise around T from v. Orient $e \in T$ away from v and $e \notin T$ "opposite" to the tour.



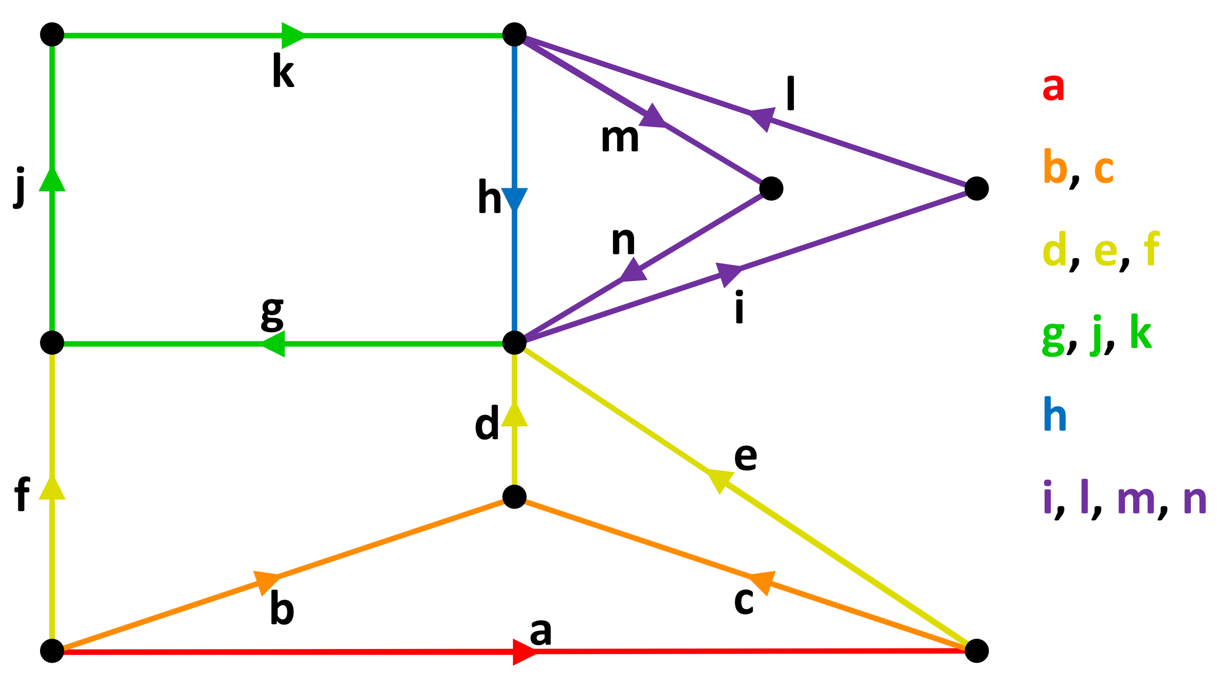
Proposition

Planar Bernardi bijections are topological bijections.

Enumeration of Activity Classes

Fix $<$ on E. For an orientation $\mathcal{O}$, let $e_1 < \ldots < e_\iota$ (resp. $e'_1 < \ldots < e'_\epsilon$) be the *internally (resp. externally) active* elements, i.e., each e_i (resp. e'_j) is the minimal element of some signed cocircuit (resp. circuit) compatible with $\mathcal{O}$.

For $k = 1, 2, \ldots, \iota$, denote by F_k the union of all signed cocircuits compatible with $\mathcal{O}$ whose minimal elements are at least e_k; dually construct $F'_1, \ldots, F'_\epsilon$. The *activity class* of $\mathcal{O}$ consists of all orientations obtained from reversing any union of components from $F_\iota, F_{\iota-1} \setminus F_\iota, \ldots, F_1 \setminus F_2; F'_\epsilon, F'_{\epsilon-1} \setminus F'_\epsilon, \ldots, F'_1 \setminus F'_2$.



Proposition (Gioan–Las Vergnas 2005; BSY 2018+)

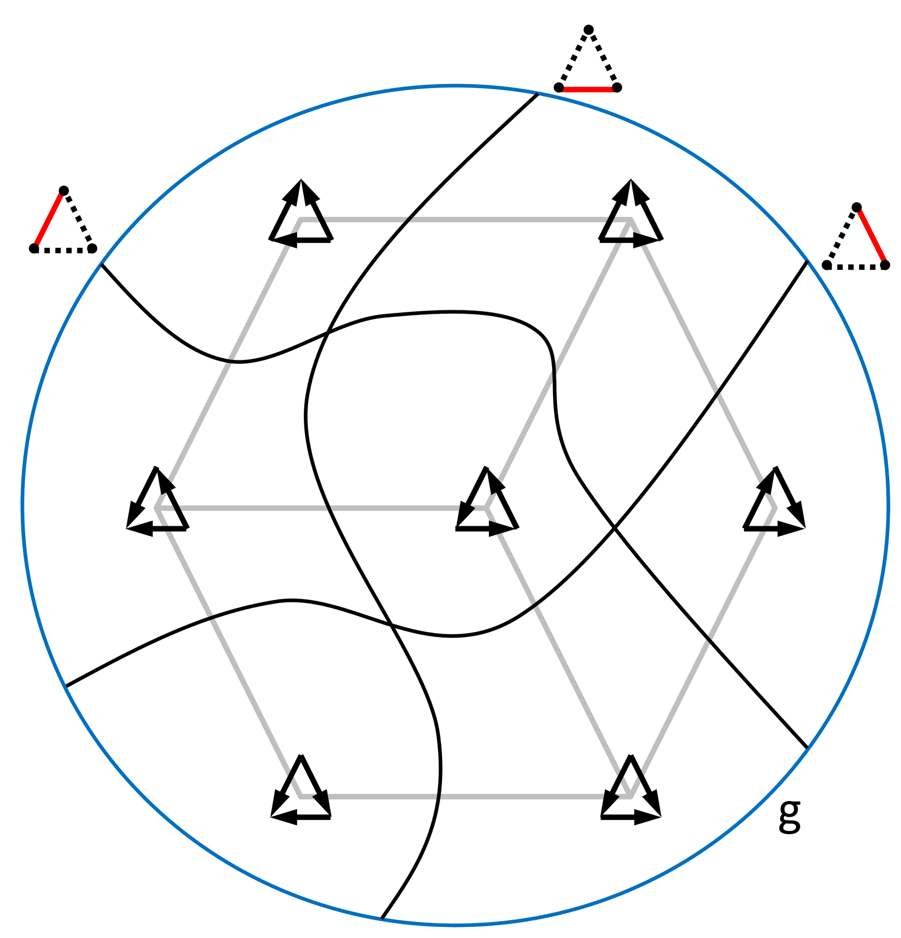
The number of activity classes of an oriented matroid equals the number of bases.

Proof: When σ,σ^* are induced by the same lexicographic data, (σ,σ^*)-compatible orientations form a system of representatives of the set of activity classes.

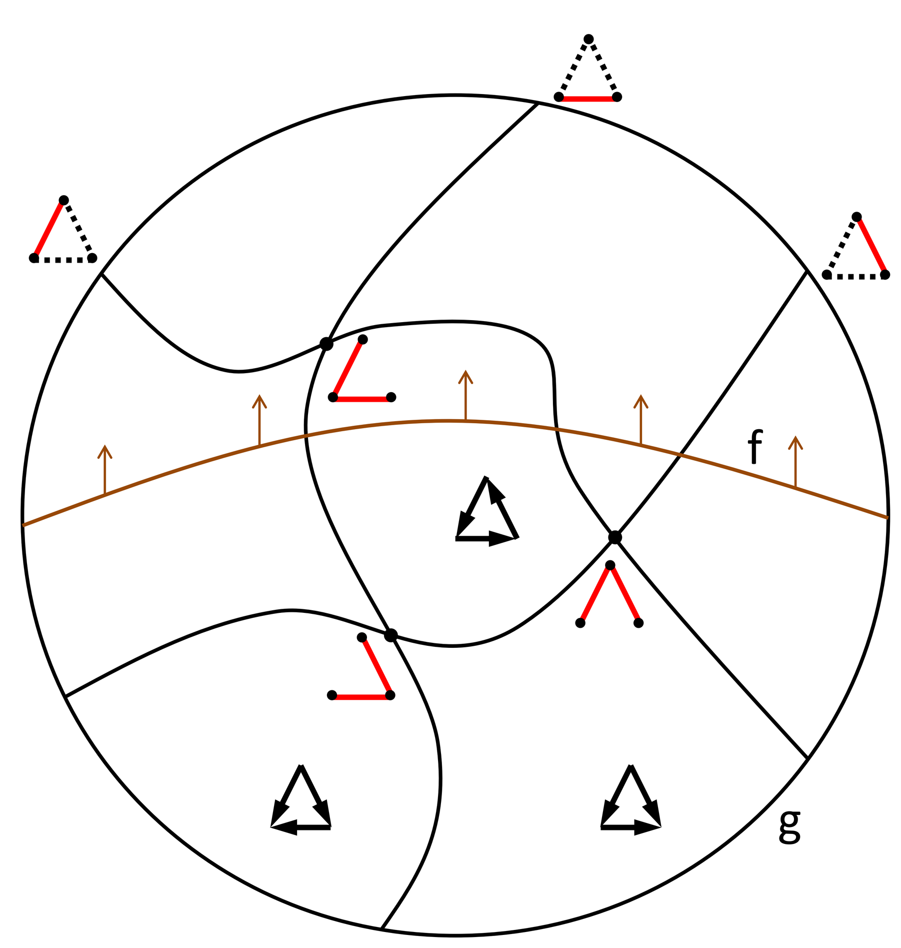
Proof of the Theorem (Sketch)

Main Ingredient: The *Oriented Matroid Program* $(\widetilde{M'}, g, f)$.

Step 1: $\widetilde{M}$ can be represented by an *affine pseudosphere arrangement* with g as the *hyperplane at infinity*. The regions are labeled by (σ-compatible) orientations of M.



Step 2: Using $f \in M'$ as an *objective functional*, each region has an optimum with respect to f. The regions whose optima are *bounded* (not lying on g) correspond precisely to (σ,σ^*)-compatible orientations. Each such optimum is a vertex, which is the intersection of pseudospheres that form a basis of M.



Step 3: The optimisation map yields a bijection, verify that it is the same as β_{σ,σ^*}.

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