Some identities involving second kind Stirling numbers of types B and D

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Two famous identities

Euler-Stirling identity

For all non-negative integers $n \geq r$, we have

$$S(n,r) = \frac{1}{r!}\sum_{k=0}^{r} A(n,k)\binom{n-k}{r-k}, \tag{1}$$

where: $S(n,r)$ = Stirling number of second kind, counting the number of partitions of $\{1,\dots,n\}$ in r blocks,
$A(n,k)$ = number of permutations $\pi \in S_n$ having $k+1$ descents (where i is a *descent* if $\pi(i) > \pi(i+1)$).

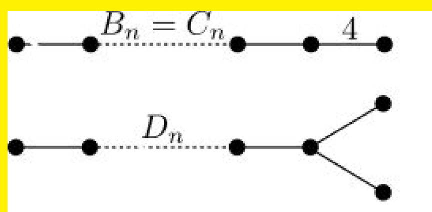
Stirling number as coefficients of falling factorials

Let $[x]_k := x(x-1)\cdots(x-k+1)$ be the *falling factorial of degree* k and $[x]_0 := 1$.
Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then we have

$$x^n = \sum_{k=0}^{n} S(n,k)[x]_k. \tag{2}$$

Main goal: Generalizations to Coxeter groups of types B and D and to the colored permutation groups $G_{r,n}$

$B_n = \mathbb{Z}_2^n \rtimes S_n$; $D_n = \{\pi \in B_n \mid \mathrm{neg}(\pi) \equiv 0 \bmod 2\}$; $G_{r,n} = \mathbb{Z}_r^n \rtimes S_n$



Preliminaries

Eulerian numbers of types B and D

$$\mathrm{Des}_B(\beta) = \{i \in [0,n-1] \mid \beta(i) > \beta(i+1)\},$$

where $\beta(0) := 0$ (we use the usual order on the integers).
In particular, $0 \in \mathrm{Des}_B(\beta)$ is a descent if and only if $\beta(1) < 0$.

$$\mathrm{des}_B(\beta) := |\mathrm{Des}_B(\beta)|.$$

$$A_B(n,k) := |\{\beta \in B_n \mid \mathrm{des}_B(\beta) = k\}|.$$

B_n-partitions, D_n-partitions, $G_{r,n}$-partitions and Stirling numbers of second kind

A *B_n-partition* is a set partition λ of $[\pm n]$ into blocks such that the following conditions are satisfied:

- There exists at most one block satisfying $-C = C$, called the *zero-block*: $C = \{\pm i \mid i \in S\} \subseteq [\pm n]$ for some $S \subseteq [n]$.
- If $C \in \lambda$ is a block in the partition λ, then $-C \in \lambda$ as well.

A *D_n-partition* is a B_n-partition such that the zero-block, if exists, contains at least two positive elements.

Example:
(a) $\{\{1,-3,6,8,-9\},\{-1,3,-6,-8,9\},\underbrace{\{2,-2\}}_{\text{zero block}},\{4,5,-7\},\{-4,-5,7\}\}$
is a B_n-partition, which is not a D_n-partition.
(b) $\{\{1,-3,6,8,-9\},\{-1,3,-6,-8,9\},\underbrace{\{2,-2,4,-4\}}_{\text{zero block}},\{5,-7\},\{-5,7\}\}$
is a D_n-partition.

Denote by $S_B(n,r)$ (*resp.* $S_D(n,r)$) the number of B_n- (*resp.* D_n-)partitions having exactly r pairs of nonzero blocks, which are called *Stirling numbers (of the second kind) of type B (resp. type D)*. They appear as sequences A039755 and A039760 in OEIS.

A B_n-partition (or D_n-partition) is called *ordered* if the set of blocks is totally ordered and the following conditions are satisfied:

- If the zero-block exists, then it appears as the first block.
- For a non-zero block C, the blocks C and $-C$ are adjacent.

A *$G_{r,n}$-partition* is a set partition of
$\Sigma = \{1,2,\dots,n,1^{[1]},2^{[1]},\dots,n^{[1]},\dots,1^{[r-1]},2^{[r-1]},\dots,n^{[r-1]}\}$
into blocks such that the following conditions are satisfied:

- There exists at most one *zero-block* satisfying $C^{[1]} = \{x^{[i+1]} \mid x^{[i]} \in C\} = C$.
- If C appears as a block in the partition λ, then $C^{[1]} \in \lambda$ as well.

Two blocks C_1 and C_2 will be called *equivalent* if there is a natural number $t \in \mathbb{N}$ such that $C_1 = C_2^{[t]} = \{x^{[i+t]} \mid x^{[i]} \in C\}$.
The number of $G_{r,n}$-partitions with r non-equivalent nonzero blocks is denoted by $S_m(n,r)$.
Example for a $G_{3,4}$-partition:

$$\left\{\underbrace{\left\{1,1^{[1]},1^{[2]},2,2^{[1]},2^{[2]}\right\}}_{\text{zero block}},\{3,4^{[1]}\},\{3^{[1]},4^{[2]}\},\{3^{[2]},4\}\right\}.$$

Euler-Stirling

Generalization of Equation (1) to type B

For all non-negative integers $n \geq r$, we have:

$$2^r r! \cdot S_B(n,r) = \sum_{k=0}^{r} A_B(n,k)\binom{n-k}{r-k}.$$

Proof's idea for type B

L.H.S. = Number of ordered set partitions of $[\pm n]$ of type B.
R.H.S. = Weighted sum of numbers of signed permutations classified by their descents.

Euler-Stirling (Cont.)

Proof's idea for type B (cont.)

Let $\pi \in B_n$ with $\mathrm{des}_B(\pi) = k$ be written in complete notation:

$$\pi = \left[\begin{array}{ccccc|ccccc} -5 & -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5 \\ -2 & 3 & 5 & -4 & -1 & 1 & 4 & -5 & -3 & 2 \end{array}\right].$$

Divide the negative part into blocks by putting separators after every descent and reflect these separators to the positive part.
In our example:

$$\pi = [\,-2\ 3\ 5 \mid -4\ -1 \mid 1\ 4 \mid -5\ -3\ 2\,].$$

Perform the following two steps:

1. If $\pi(-1)$ and $\pi(1)$ are in the same block (the zero-block), then move this block to the beginning.
2. For each non-zero block B contained in the negative part of π, locate the block $-B$ right after it.

If $r = k$, then we have associated to the signed permutation π an ordered set partition of type B and we are done.
If $r > k$, refine the partition by simultaneously splitting pairs of blocks of the form B and $-B$ (where $B \neq -B$), or by splitting a zero-block.
Example (cont.) $\beta = [\,1,4 \mid -5,-3,2 \mid\,] \in B_5$ produces the ordered B_5-partition with one pair of nonzero blocks $[\{\pm1,\pm4\},\{-5,-3,2\},\{5,3,-2\}]$, and exactly $\binom{4}{1}$ ordered B_5-partitions with two pairs of nonzero blocks, namely:

$$[\{1,4\},\{-1,-4\},\{-5,-3,2\},\{5,3,-2\}],$$
$$[\{\pm1\},\{4\},\{-4\},\{-5,-3,2\},\{5,3,-2\}],$$
$$[\{\pm1,\pm4\},\{-5\},\{5\},\{-3,2\},\{3,-2\}],$$
$$[\{\pm1,\pm4\},\{-5,-3\},\{5,3\},\{2\},\{-2\}],$$

obtained by placing one artificial separator before entries $1,2,4$ and 5, respectively. The other ordered partitions coming from β with more blocks are obtained similarly.

Generalization of Equation (1) to type D and its idea of proof

For all non-negative integers $n \geq r$, with $n \neq 1$, we have:

$$2^r r! \cdot S_D(n,r) = \left[\sum_{k=0}^{r} A_D(n,k)\binom{n-k}{r-k} + n \cdot 2^{n-1}(r-1)! \cdot S(n-1,r-1)\right]$$

where $S(n-1,r-1)$ is the usual Stirling number of second kind.

Proof's idea: The proof for type D is a bit more tricky. The basic idea is the same as in type B, except for that when a zero block with only one pair is liable to be obtained, i.e. the permutation has a descent **after** the first digit, but not **before** that digit, we switch to a permutation of $B_n - D_n$.
As a result, some of the D_n-partitions are not obtained, so we count them separately.

Falling factorials

Generalization of Equation (2) to type B (Remmel-Wachs, Bala)

Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then we have:

$$x^n = \sum_{k=0}^{n} S_B(n,k)[x]_k^B,$$

where $[x]_k^B := (x-1)(x-3)\cdots(x-2k+1)$ and $[x]_0^B := 1$.

Our combinatorial proof's idea for type B (suggested to us by V. Reiner)

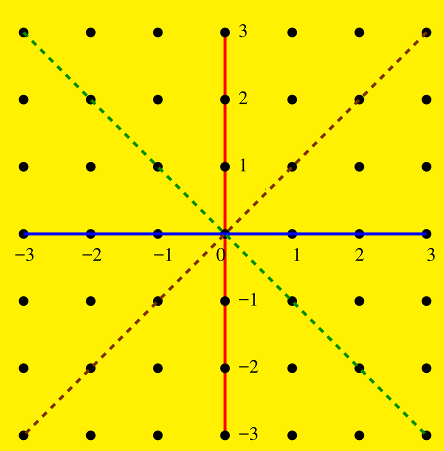
It is sufficient to prove the identity for odd integers $x = 2m+1$:
L.H.S counts lattice points of the cube $[-m,m]^n \cap \mathbb{Z}^n$.
R.H.S. exploits B_n-partitions to count these points using the maximal intersection subsets of hyperplanes the points lay on.
Example: The B_6-partition

$$\lambda = \{\{1,-2,4\},\{-1,2,-4\},\{3,-5\},\{-3,5\},\{6,-6\}\}$$

corresponds to the subspace:

$$\{x_1 = -x_2 = x_4\} \cap \{x_3 = -x_5\} \cap \{x_6 = 0\}.$$

Example: Let $n = 2$ and $m = 3$, so $x = 2m+1 = 7$.



$k = 0$: The only B_2-partition with 0 non-zero blocks is $\lambda_0 = \{\{1,-1,2,-2\}\}$ corresponding to the subspace $\{x_1 = x_2 = 0\}$, containing only $(0,0)$.

$k = 1$: We have four B_2-partitions, two of them contain a zero-block:

$$\lambda_1 = \{\{1,-1\},\{2\},\{-2\}\} \quad \mapsto \quad \{(x_1,x_2) \mid x_1 = 0\}$$
$$\lambda_2 = \{\{2,-2\},\{1\},\{-1\}\} \quad \mapsto \quad \{(x_1,x_2) \mid x_2 = 0\}$$

and two of them do not:

$$\lambda_3 = \{\{1,2\},\{-1,-2\}\} \quad \mapsto \quad \{(x_1,x_2) \mid x_1 = x_2\}$$
$$\lambda_4 = \{\{1,-2\},\{-1,2\}\} \quad \mapsto \quad \{(x_1,x_2) \mid x_1 = -x_2\}.$$

Each of these hyperplanes contains 6 points (w/o the origin).

Falling factorials (Cont.)

Combinatorial proof (cont.)

The rest of the points are counted in the case of $k = n = 2$ pairs of non-zero blocks:
$k = 2$: The single B_2-partition:

$$\lambda_5 = \{\{1\},\{-1\},\{2\},\{-2\}\} \quad \mapsto \quad \{(x_1,x_2) \mid x_1 \neq \pm x_2 \neq 0\}$$

which are the lattice points not lying on any hyperplane.

Falling factorial for type D

$$[x]_k^D := \begin{cases} 1, & k = 0; \\ (x-1)(x-3)\cdots(x-(2k-1)), & 1 \leq k < n; \\ (x-1)(x-3)\cdots(x-(2n-3))(x-(n-1)), & k = n. \end{cases}$$

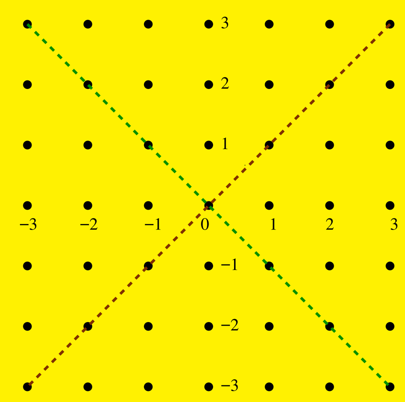
Generalization of Equation (2) to type D

For all $n \in \mathbb{N}$ and $x \in \mathbb{R}$:

$$x^n = \sum_{k=0}^{n} S_D(n,k)[x]_k^D + n\left((x-1)^{n-1} - [x]_{n-1}^D\right).$$

Proof's idea for type D

Let $n = 2$ and $m = 3$, so $x = 2m+1 = 7$.



$k = 0$: we have exactly one D_2-partition $\lambda_0 = \{\{1,-1,2,-2\}\}$ which counts only the lattice point $(0,0)$.

$k = 1$: we have only two D_2-partitions:

$$\lambda_1 = \{\{1,2\},\{-1,-2\}\} \quad \mapsto \quad \{(x_1,x_2) \mid x_1 = x_2\}$$
$$\lambda_2 = \{\{1,-2\},\{-1,2\}\} \quad \mapsto \quad \{(x_1,x_2) \mid x_1 = -x_2\}$$

$k = 2$: There is a single D_2-partition:

$$\lambda_3 = \{\{1\},\{-1\},\{2\},\{-2\}\} \quad \mapsto \quad \{(x_1,x_2) \mid x_1 \neq \pm x_2\}$$

Now, the value 0 can also appear (different from type B, since the axes were not counted in step $k = 1$ of type D).
These are all the lattice points which do not lie on the diagonals.

The missing lattice points for $n \geq 3$

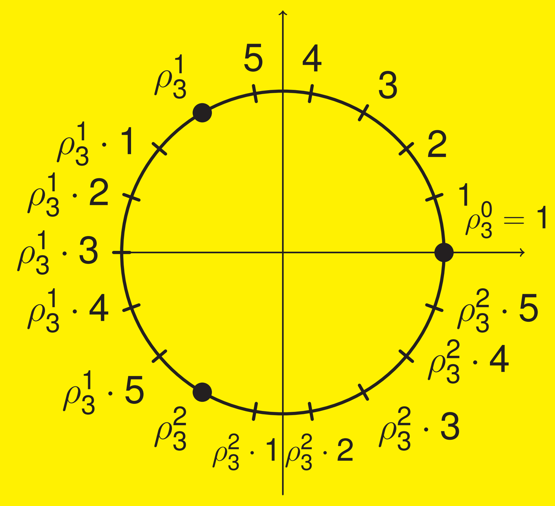
When $n \geq 3$, there are points which are not counted. They have the form (x_1,x_2,x_3), such that exactly one of their coordinates is 0 and the other two share the same absolute value. e.g. the points $(0,2,2)$ and $(0,2,-2)$ are not counted.
The number of such missing lattice points (which is the second summand in the R.H.S. for $n = 3$) is: $3 \cdot 6^2 - 3 \cdot 6 \cdot 4 = 36$.

Generalization of Equation (2) to colored permutation groups $G_{r,n}$

Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then we have: $x^n = \sum_{k=0}^{n} S_m(n,k)[x]_k^m$.

Sketch of the proof:

- Divide the unit circle S^1 according to the mth roots of unity: $1,\rho_m,\rho_m^2,\dots,\rho_m^{m-1}$. This divides the circle into m arcs.
- In each arc, locate t points in equal distances from each other.
- We get $x = mt+1$ points on the unit circle, including the point $(1,0)$.



- Consider the n-dimensional torus $(S^1)^n = S^1 \times \cdots \times S^1$ with x^n *lattice points on it*.
  The same arguments will apply, when we interpret the $G_{r,n}$-partitions as intersections of subsets of hyperplanes in the generalized hyperplane arrangement $\mathcal{G}_{m,n}$ for the colored permutation group:

$$\mathcal{G}_{m,n} := \{\,\{x_i = \rho_m^k x_j\} \mid 1 \leq i < j \leq n, 0 \leq k < m\}$$
$$\cup\ \{\,\{x_i = 0\} \mid 1 \leq i \leq n\},$$

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