Polynomial invariant and reciprocity theorem on the Hopf monoid of hypergraphs

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Motivation

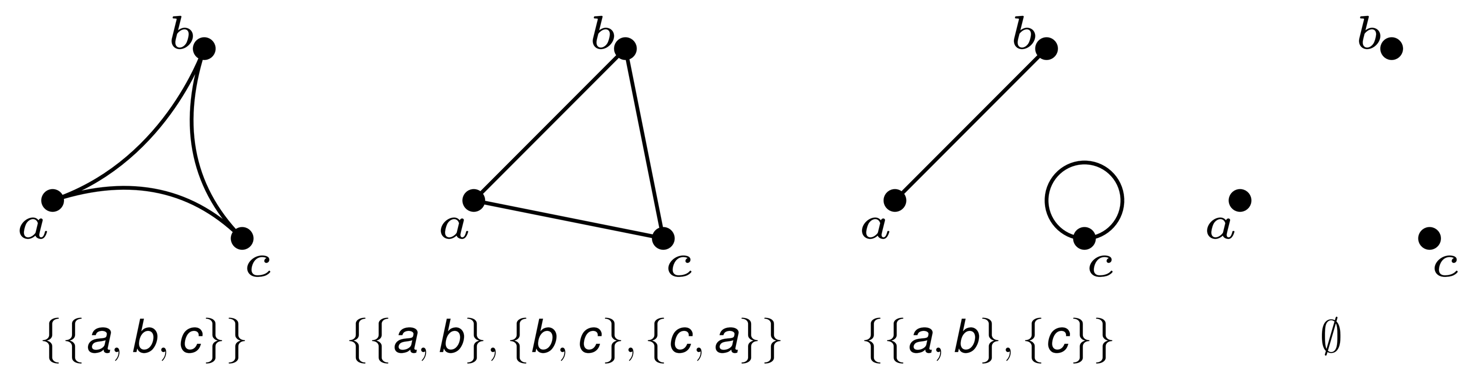
In 2017 Aguiar and Ardila published a seminal paper on Hopf monoids ([1]) where they give a theorem to build polynomial invariants and reciprocity theorems on combinatorial objects. In the same paper they define a Hopf monoid on hypergraphs and asked for an interpretation of the polynomial invariant obtained by their theorem on this Hopf monoid. Our work [2] is an answer to this question. While the obtained invariant and its reciprocity theorem are present in Postnikov's lecture [3] within a poytope context, our approach is a different one.

Hopf Monoid

A *Hopf monoid* M is given by the data of:

- for each finite set I, a vector space $P[I]$,
- for each bijection $\sigma : I \to J$, a linear map $P[\sigma] : P[I] \to P[J]$, such that $P[\tau \circ \sigma] = P[\tau] \circ P[\sigma]$ and $P[\mathrm{id}] = \mathrm{id}$.

Example: If HG is the Hopf monoid of hypergraphs then $HG[I]$ is the vector space spanned by hypergraphs over I and $HG[\sigma]$ is a relabeling of the vertices.



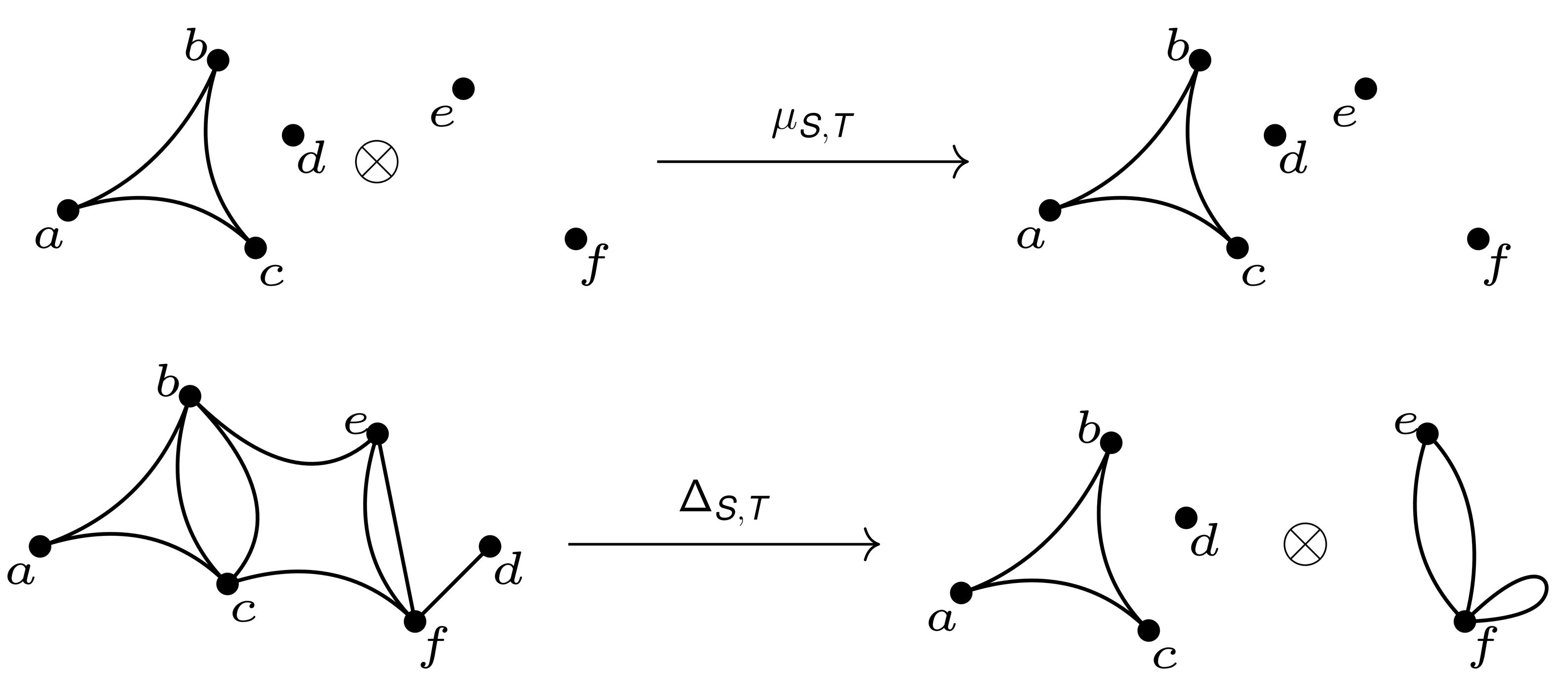
Elements of $HG[\{a,b,c\}]$ basis.

Furthermore M must come with, for every set $I = S \sqcup T$:

- a product $\mu_{S,T} : M[S] \otimes M[T] \to M[I]$,
- a co-product $\Delta_{S,T} : M[I] \to M[S] \otimes M[T]$.

These two operations satisfying diverse axioms.

Example: A possible product and coproduct on HG are the one acting as in the following example. For $S = \{a,b,c,d\}$ and $T = \{e,f\}$:



A *Hopf monoid character* $\zeta : M \to \mathbb{k}$ is a collection of linear forms $\zeta_I : M[I] \to \mathbb{k}$ such that for every $I = S \sqcup T$ and $x \in M[S]$ and $y \in M[T]$ we have $\zeta_I(xy) = \zeta_S(x)\zeta_T(y)$.

Example: $\zeta^d_I(H) = \begin{cases} 1 \text{ if } H \text{ is discrete} \\ 0 \text{ else} \end{cases}$

Polynomial invariant [1]

Let M be a Hopf monoid, ζ a character and $x \in M[I]$. Then

$$\chi_I(x)(n) = \sum_{S_1 \sqcup \dots \sqcup S_n = I} \zeta_{S_1} \otimes \dots \otimes \zeta_{S_n} \circ \Delta_{S_1,\dots,S_n}(x)$$

is a polynomial in n such that $\chi_I(xy) = \chi(x)\chi(y)$ and $\chi_I(x)(1) = \zeta_I(x)$.

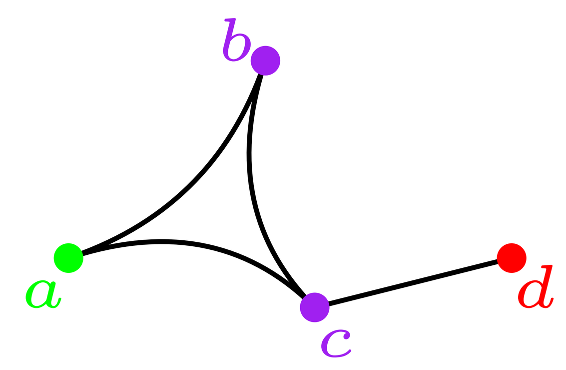
Question

What is the value of χ_I over both positive and negative integers for the Hopf monoid HG and the character ζ^d ?

Colouring, orientation and compatibility

A *colouring* of H with $[n]$ is a function from the vertices to $[n]$: $c : I \to [n]$. Let $e \in H$. Then $v \in e$ is *maximal* in e (for c) if v is of maximal color in e.

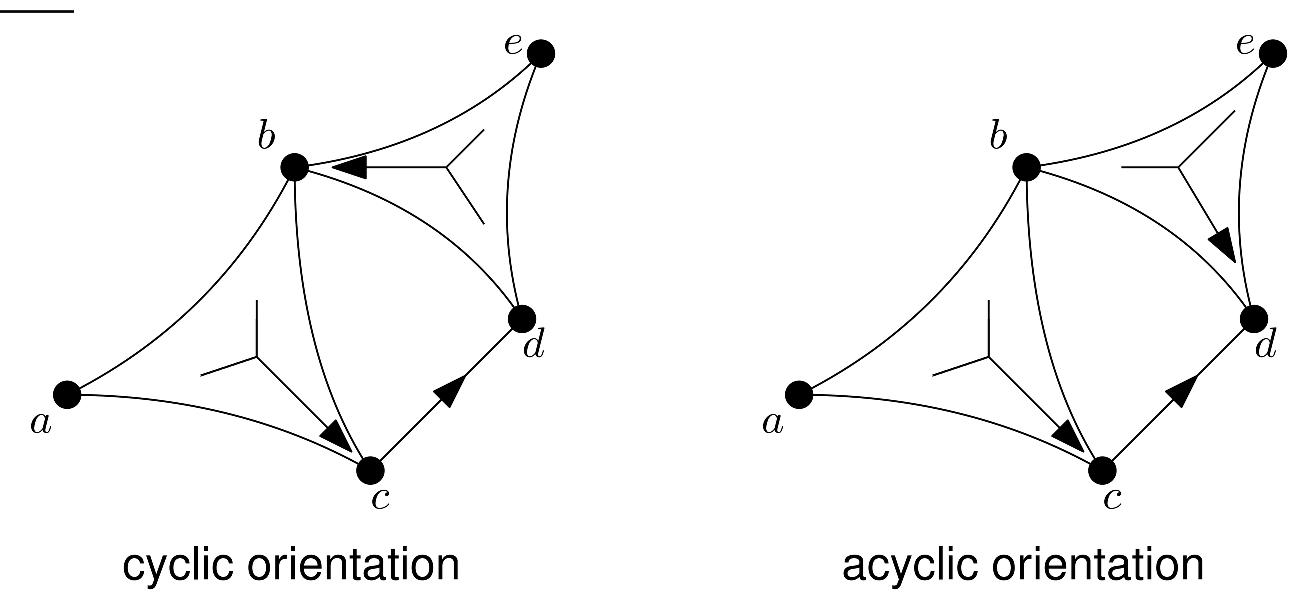
Example: Colouring with $\{1, 2, 3\}$.



b and c are maximal in $\{a,b,c\}$, d is maximal in $\{c,d\}$.

An *orientation* of H is a function $f : H \to I$ which associates to every edge $e \in H$ an *exit* $f(e) \in e$. A *cycle* in f is a sequence $e_1, \dots, e_k$ of edges such that $f(e_1) \in e_2 \setminus f(e_2), \dots, f(e_k) \in e_1 \setminus f(e_1)$. We note $\mathcal{A}_H$ the set of acyclic orientations of H.

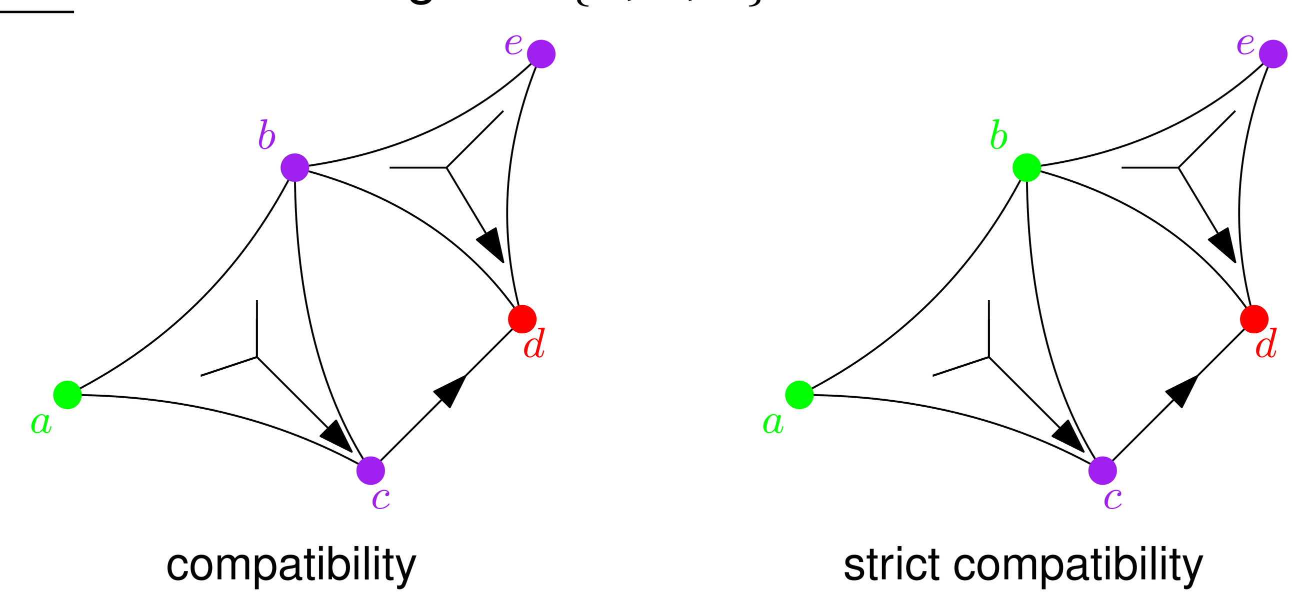
Example:



Two orientations of a hypergraph.

A colouring and an orientation are said to be *compatible* if the exits are maximal within their edges. They are said to be *strictly compatible* if the exits are the only maximal vertex within their edge.

Example: With colouring with $\{1, 2, 3\}$.



Pairs of compatible colourings and orientations.

Hypergraph polynomial invariant

Let I be a set, H a hypergraph over I and n an integer.

Then $\chi_I(H)(n)$ is the number of colourings of H with $[n]$ such that every edge has only one maximal vertex. This is also the number of strictly compatible pairs of acyclic orientations and colourings with $[n]$ of H.

Furthermore $(-1)^{|I|}\chi_I(H)(-n)$ is the number of compatible pairs of acyclic orientations and colourings with $[n]$ of H. In particular, $(-1)^{|I|}\chi_I(H)(-1)$ is the number of acyclic orientations of H.

References

[1] Marcelo Aguiar and Federico Ardila. Hopf monoids and generalized permutahedra, 2017. arXiv:1709.07504.

[2] Jean-Christophe Aval, Théo Karaboghossian, and Adrian Tanasa. The hopf monoid of hypergraphs and its sub-monoids: basic invariant and reciprocity theorem, 2018. arXiv:1806.08546.

[3] Alexander Postnikov. Topics in Combinatorics: Polytopes, 2016. http://math.mit.edu/ apost/courses/18.218 2016.

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