SKEW POLYNOMIALS AND EXTENDED SCHUR FUNCTIONS

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ABSTRACT

Using Kohnert's algorithm, we associate a polynomial to any cell diagram in the positive quadrant, simultaneously generalizing Schubert polynomials and GL_n Demazure characters. We survey properties of these Kohnert polynomials and their stable limits, which are quasisymmetric functions. As a first application, we introduce and study two new bases of Kohnert polynomials, one of which stabilizes to the skew-Schur functions and is conjecturally Schubert-positive, the other stabilizes to a new basis of quasisymmetric functions that contains the Schur functions.

QUESTION

What choices of cell diagrams of weight $\mathbf{a}$ yield interesting

- Kohnert bases of polynomials?
- quasisymmetric stable limits?

STABLE LIMITS

The ***monomial slide polynomials*** [1] form a basis of polynomials. Their ***stable limits*** are the monomial quasisymmetric functions [4].

Theorem 1. *[3] Kohnert polynomials expand nonnegatively in the monomial slide basis.*

Define the stable limit $\mathcal{K}_D$ of $\mathfrak{K}_D$ by

$$\mathcal{K}_D = \lim_{m\to\infty} \mathfrak{K}_{0^m \times D}$$

where $0^m \times D$ is D shifted m rows upwards.

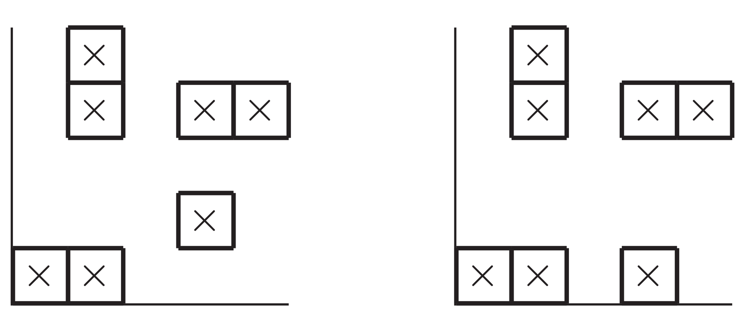
Corollary 1. *[3] The stable limits of Kohnert polynomials are quasisymmetric (and expand nonnegatively in monomial quasisymmetric functions).*

Theorem 2. *[3] The stable limits of Kohnert polynomials expand nonnegatively in fundamental quasisymmetric functions.*

SOUTHWEST DIAGRAMS

Definition 2. A diagram D is ***southwest*** if whenever $(r_2, c_1), (r_1, c_2) \in D$, where $r_1 < r_2$ and $c_1 < c_2$, then (r_1, c_1) is also in D.

The diagram on the left is not southwest, while the diagram on the right is.



Conjecture 1. *Given a southwest diagram D, the Kohnert polynomial $\mathfrak{K}_D$ expands non-negatively into key polynomials.*

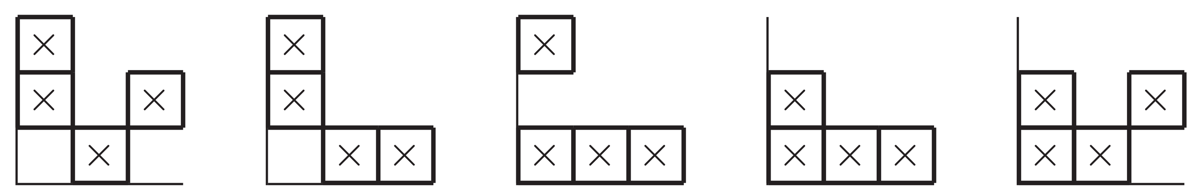
KOHNERT POLYNOMIALS AND KOHNERT BASES

A ***Kohnert move*** [5] on a diagram moves the rightmost cell of a given row to the first available position below, jumping over other cells in its way as needed. Given a diagram D, let $\mathrm{KD}(D)$ denote the set of all diagrams that can be obtained by applying a series of Kohnert moves to D.

Definition 1. The ***Kohnert polynomial indexed by*** D is

$$\mathfrak{K}_D = \sum_{T \in \mathrm{KD}(D)} x^{\mathrm{wt}(T)}.$$

For example, below is a cell diagram D (leftmost) and all Kohnert diagrams of D.



Then $\mathfrak{K}_D = x^{121} + x^{211} + x^{301} + x^{310} + x^{220}$.

Any **choice** of diagram of weight $\mathbf{a}$, i.e., with $\mathbf{a}_i$ cells in row i, gives a basis of the polynomial ring. We call bases arising in this manner ***Kohnert bases***. For example, Kohnert proved [5] that when D is left-justified, $\mathfrak{K}_D$ is a ***key polynomial*** (Demazure character).

Theorem 3. *[3] Given any set of diagrams $\{D_{\mathbf{a}}\}$, one of weight $\mathbf{a}$ for each weak composition $\mathbf{a}$, the corresponding Kohnert polynomials $\{\mathfrak{K}_{D_{\mathbf{a}}}\}$ form a basis of the polynomial ring.*

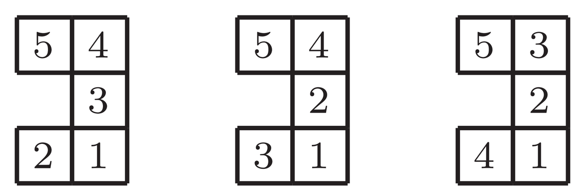
LOCK POLYNOMIALS AND EXTENDED SCHUR FUNCTIONS

The ***lock diagram*** $\mathbb{L}(\mathbf{a})$ is the **right-justified** diagram of weight $\mathbf{a}$. the corresponding Kohnert basis is the ***lock polynomials***. Their stable limits are the ***extended Schur functions*** $\mathcal{E}_\alpha$.

Theorem 4. *[3] Extended Schur functions are a basis of quasisymmetric functions that contains the Schur functions.*

Definition 3. [3] The ***standard extended tableaux of shape*** α, denoted $\mathrm{SET}(\alpha)$, are the bijective fillings of $\mathbb{L}(\alpha)$ with $1, 2, \ldots, n$ such that entries in each row decrease from left to right and entries in each column decrease from top to bottom.

For example, the standard extended tableaux of shape $(2, 1, 2)$ are

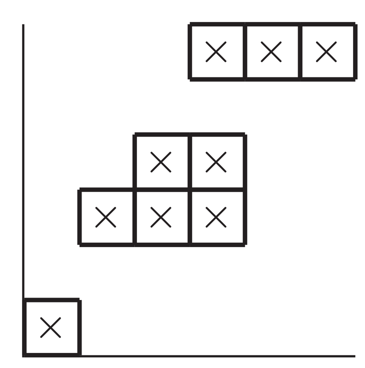


Theorem 5. *[3] For α a (strong) composition, we have*

$$\mathcal{E}_\alpha(X) = \sum_{T \in \mathrm{SET}(\alpha)} F_{\mathrm{Des}(T)}(X).$$

SKEW POLYNOMIALS AND SKEW-SCHUR FUNCTIONS

Another Kohnert basis is given by the ***skew polynomials*** $\{\mathfrak{K}_{\mathbb{S}(\mathbf{a})}\}$; the Kohnert polynomials of the ***skew diagrams*** $\mathbb{S}_a$. The skew diagram $\mathbb{S}(1, 0, 3, 2, 0, 3)$ is shown below.



Theorem 6. *[3] Skew polynomials expand nonnegatively in key polynomials, and their stable limits are the skew-Schur functions.*

Skew diagrams are southwest, which fits with Conjecture 1.

Conjecture 2. *Skew polynomials expand nonnegatively into Schubert polynomials.*

REFERENCES

[1] S. Assaf and D. Searles. Schubert polynomials, slide polynomials, Stanley symmetric functions and quasi-Yamanouchi pipe dreams. *Adv. Math.*, 306:89–122, 2017.

[2] S. Assaf and D. Searles. Kohnert tableaux and a lifting of quasi-Schur functions. *J. Combin. Theory Ser. A*, 156:85–118, 2018.

[3] S. Assaf and D. Searles. Kohnert polynomials. *Experiment. Math.*, 2019. To appear.

[4] I. M. Gessel. Multipartite P-partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 1984.

[5] A. Kohnert. Weintrauben, Polynome, Tableaux. *Bayreuth. Math. Schr.*, (38):1–97, 1991. Dissertation, Universität Bayreuth, Bayreuth, 1990.

[6] V. Reiner and M. Shimozono. Key polynomials and a flagged Littlewood-Richardson rule. *J. Combin. Theory Ser. A*, 70(1):107–143, 1995.

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