A bijection between ordinary partitions and self-conjugate partitions with same disparity

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Introduction

- A **partition** $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ of n is a positive integer sequence such that $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_\ell$ and $\sum_{i=1}^{\ell}\lambda_i=n$. We denote the set of partitions of n by $\mathscr{P}(n)$, and let $p(n)=|\mathscr{P}(n)|$ and $\mathscr{P}=\cup_{n\ge0}\mathscr{P}(n)$.
- The **conjugate** $\lambda'=(\lambda'_1,\lambda'_2,\ldots,\lambda'_{\lambda_1})$ of a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_\ell)$ is the partition whose Young diagram can be obtained by the reflection of the Young diagram of λ along the main diagonal.
- A **self-conjugate** partition λ is a partition satisfying that $\lambda'=\lambda$. We denote the set of self-conjugate partitions of n by $\mathscr{SC}(n)$, and let $sc(n)=|\mathscr{SC}(n)|$ and $\mathscr{SC}=\cup_{n\ge0}\mathscr{SC}(n)$.
- A **t-core** is a partition with no hook length divisible by t, where the **hook length** is the number $h(i,j)=\lambda_i+\lambda'_j-i-j+1$ for a box (i,j) in the Young diagram of a paritions. We say that a partition is a **$(t_1,\ldots,t_p)$-core** if it is simultaneously a t_1-core, ... , a t_p-core. We use the notations $c_{(t_1,\ldots,t_p)}(n)$ for the number of $(t_1,\ldots,t_p)$-cores of n and $sc_{(t_1,\ldots,t_p)}(n)$ for the number of self-conjugate $(t_1,\ldots,t_p)$-cores of n.

Goals.

We give a bijection between the set of ordinary partitions and the set of self-conjugate partitions with same disparity.
Also, we show a relation between hook lengths of a partition and the corresponding self-conjugate partition via the bijection. Using this bijection, we have the followings:

- Combinatorial proofs of the identities

$$\sum_{n=0}^{\infty}sc(n)q^n=\left(\sum_{n=0}^{\infty}p(n)q^{4n}\right)\left(\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\right)\quad\text{and}\quad\sum_{n=0}^{\infty}sc_{2t}(n)q^n=\left(\sum_{n=0}^{\infty}c_t(n)q^{4n}\right)\left(\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\right).\tag{1}$$

- A generalization of (1), an identity involving the generating function of the number $sc_{(2t_1,2t_2,\ldots,2t_p)}(n)$ and the generating function of the number $c_{(t_1,t_2,\ldots,t_p)}(n)$, see Corollary 4.
- Immediate consequences of the results of Anderson and Wang;

$$c_{(t_1,t_2)}=\frac{1}{t_1+t_2}\binom{t_1+t_2}{t_1}\quad\text{and}\quad c_{(n,n+d,n+2d)}=\frac{1}{n+d}\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n+d}{i,i+d,n-2i},\tag{2}$$

so that we give new combinatorial interpretations for the Catalan number C_n and the Motzkin number M_n in terms of self-conjugate simultaneous core partitions, see Corollary 5.

New classification of $\mathscr{SC}(n)$

For a self-conjugate partition λ, entries of the set $D(\lambda)$ of main diagonal hook lengths of λ are all odd and distinct. Hence, $D(\lambda)$ can be partitioned into

$$D_1(\lambda)=\{\delta_i\in D(\lambda)\mid\delta_i\equiv1\pmod 4\}\quad\text{and}\quad D_3(\lambda)=\{\delta_i\in D(\lambda)\mid\delta_i\equiv3\pmod 4\}.$$

For a nonnegative integer m, we define a subset of $\mathscr{SC}(n)$ by

$$\mathscr{SC}^{(m)}(n)=\left\{\lambda\in\mathscr{SC}(n)\ :\ |D_1(\lambda)|-|D_3(\lambda)|=(-1)^{m+1}\left\lceil\frac{m}{2}\right\rceil\right\}$$

so that $\mathscr{SC}(n)=\cup_{m\ge0}\mathscr{SC}^{(m)}(n)$.

Self-conjugate partitions with same disparity

We define the **disparity** of a partition λ to be the number

$$\mathrm{dp}(\lambda)=|\{(i,j)\in\lambda\mid h(i,j)\text{ is odd}\}|-|\{(i,j)\in\lambda\mid h(i,j)\text{ is even}\}|.$$

We note that for any parition λ, its disparity is always of the form $m(m+1)/2$ for a positive integer m.

Proposition 1. For a nonnegative integer m, if $\lambda\in\mathscr{SC}^{(m)}$, then its disparity $\mathrm{dp}(\lambda)$ is $\frac{m(m+1)}{2}$ so that

$$\mathscr{SC}^{(m)}=\left\{\lambda\in\mathscr{SC}\mid\mathrm{dp}(\lambda)=\frac{m(m+1)}{2}\right\}.$$

A bijection between partitions and self-conjugate partitions with same parity

For a self-conjugate partition λ, the **diagonal sequence pair** $((a_1,a_2,\ldots,a_r),(b_1,b_2,\ldots,b_s))$ of λ is defined to be a pair of sequences (a_i) and (b_j) satisfying that $a_1>a_2>\cdots>a_r\ge0$, $b_1>b_2>\cdots>b_s\ge1$,

$$D_1(\lambda)=\{4a_1+1,4a_2+1,\ldots,4a_r+1\}\quad\text{and}\quad D_3(\lambda)=\{4b_1-1,4b_2-1,\ldots,4b_s-1\}.$$

Bijection $\phi_n^{(m)}\ :\ \mathscr{SC}^{(m)}(4n+m(m+1)/2)\ \to\ \mathscr{P}(n)$

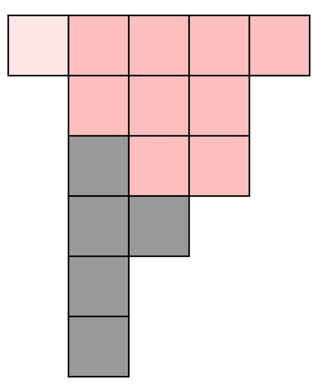
Let $\lambda\in\mathscr{SC}^{(m)}(4n+m(m+1)/2)$ be the self-conjugate partition with a diagonal sequence pair $((a_1,\ldots,a_r),(b_1,\ldots,b_s))$ so that $r-s+(-1)^m\lceil m/2\rceil=0$ and $4(\sum_{i=1}^{r}a_i+\sum_{j=1}^{s}b_j)+r-s=4n+m(m+1)/2$. We define $\phi_n^{(m)}(\lambda)$ by the partition $\mu=(\mu_1,\ldots,\mu_\ell)$ of n such that

$$\mu_i=a_i+i+s-r\quad\text{for}\quad i\le r,\qquad\text{and}\qquad(\mu_{r+1},\ldots,\mu_\ell)\text{ is the conjugate of the partition }\gamma=(b_1-s,b_2-s+1,\ldots,b_s-1).$$

Examples of the bijection

Example 1. Let $\mu=(4,3,3,2,1,1)\in\mathscr{P}(14)$.

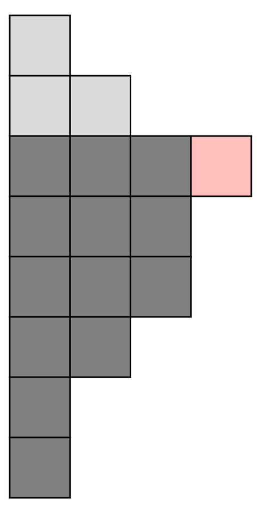
Odd case : For $m=3$:



$62=14\times4+\binom{4}{2}$

$\Leftrightarrow((5,3,2,0),(4,1))$
$D_1(\lambda)=\{21,13,9,1\}$, $D_3(\lambda)=\{15,3\}$

Even case : For $m=4$:



$66=14\times4+\binom{5}{2}$

$\Leftrightarrow((1),(8,5,3))$
$D_1(\lambda)=\{5\}$, $D_3(\lambda)=\{31,19,11\}$

Example 2. We observe relations between hook lengths.

Case $\delta_1\in D_3(\lambda)$:

11	8	5	4	2	1
8	5	2	1		
5	2				
4	1				
2					
1					

$\Leftrightarrow$

4	1
2	
1	

$m=0$

Case $\delta_1\in D_1(\lambda)$:

9	8	5	3	2
8	7	4	2	1
5	4	1		
3	2			
2	1			

$\Leftrightarrow$

4	1
2	
1	

$m=1$

Relations between hook lengths

Proposition 2. For $\lambda\in\mathscr{SC}$ with $D(\lambda)=\{\delta_1,\delta_2,\ldots,\delta_d\}$, let $\bar\lambda$ be the self-conjugate partition with $D(\bar\lambda)=D(\lambda)/\{\delta_1\}$. If $\mu=(\mu_1,\mu_2,\ldots,\mu_\ell)$ (resp. $\bar\mu$) be the corresponding partition of λ (resp. $\bar\lambda$), then

$$\bar\mu=\begin{cases}(\mu_2,\mu_3,\ldots,\mu_\ell), & \text{if}\quad\delta_1\in D_1(\lambda),\\(\mu_1-1,\mu_2-1,\ldots,\mu_\ell-1), & \text{if}\quad\delta_1\in D_3(\lambda).\end{cases}$$

Theorem 3. For a self-conjugate partition λ with the disparity $m(m+1)/2$, let $\phi^{(m)}(\lambda)=\mu$. Then, for each positive integer k, we have

$$|\{(i,j)\in\lambda\mid h_\lambda(i,j)=2k\}|=2|\{(\tilde i,\tilde j)\in\mu\mid h_\mu(\tilde i,\tilde j)=k\}|.$$

Corollary 4. For a self-conjugate partition λ with the disparity $m(m+1)/2$, let $\phi^{(m)}(\lambda)=\mu$. Then λ is a $(2t_1,2t_2,\ldots,2t_p)$-core if and only if μ is a $(t_1,t_2,\ldots,t_p)$-core. Hence, we have

$$\sum_{n=0}^{\infty}sc_{(2t_1,\ldots,2t_p)}(n)q^n=\left(\sum_{n=0}^{\infty}c_{(t_1,\ldots,t_p)}(n)q^{4n}\right)\left(\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\right).$$

New interpretations of Catalan and Motzkin

Corollary 5. Let m be a nonnegative integer and n, d are positive relatively prime integers.

- The number of self-conjugate $(2n,2d)$-cores with the disparity $m(m+1)2$ is

$$sc^{(m)}_{(n,d)}=\frac{1}{n+d}\binom{n+d}{n}.$$

- The number of self-conjugate $(2n,2n+2d,2n+4d)$-cores with the disparity $\frac{m(m+1)}{2}$ is

$$sc^{(m)}_{(2n,2n+2d,2n+4d)}=\frac{1}{n+d}\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n+d}{i,i+d,n-2i}.$$

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