Enumerating involution words for signed permutations

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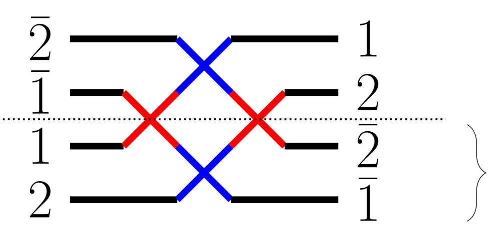
Signed permutations

- A permutation w of $[\pm n] := \{-n, \dots, -1, 1, \dots, n\}$ is *signed* if $w(-i) = -w(i)$.
- The group B_n of signed permutations of $[\pm n]$ is a Coxeter group generated by
  $$s_0 := (-1, 1) \quad \text{and} \quad s_i := (i, i+1)(-i, -i-1) \text{ for } 1 \le i < n.$$
- The *one-line notation* for $w \in B_n$ is the word $w(1)w(2)\cdots w(n)$, writing $\bar{a}$ for $-a$:
  $$w = 2\bar{1} \text{ maps } 1 \mapsto 2 \text{ and } 2 \mapsto -1, \text{ hence also } -1 \mapsto -2 \text{ and } -2 \mapsto 1.$$
- Let $\ell(w)$ denote the Coxeter length of $w \in B_n$.
- The unique longest element in B_n is $w_n^B := \bar{1}\bar{2}\bar{3}\cdots\bar{n}$
- The longest element in B_n preserving $\{1, \dots, n\}$ is $w_n^A := n \cdots 321$

Reduced words

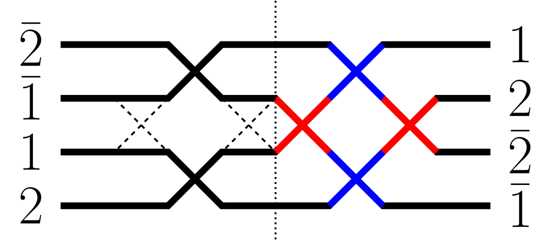
- *Reduced word* for $w \in B_n$: minimal-length word $\mathbf{a} = a_1 \cdots a_\ell$ with $w = s_{a_1} \cdots s_{a_\ell}$.
- For example, $\mathbf{a} = \mathbf{010}$ is a reduced word for $w = 2\bar{1} \in B_2$:
  $$s_0 s_1 s_0 = 12 \cdot s_0 s_1 s_0 = \bar{1}2 \cdot s_1 s_0 = 2\bar{1} \cdot s_0 = \bar{2}\bar{1}.$$
- Each word $\mathbf{a} = a_1 \cdots a_\ell$ corresponds to a wiring diagram for $w = s_{a_1} \cdots s_{a_\ell} \in B_n$, a vertically symmetric configuration of $2n$ wires connecting parallel columns:

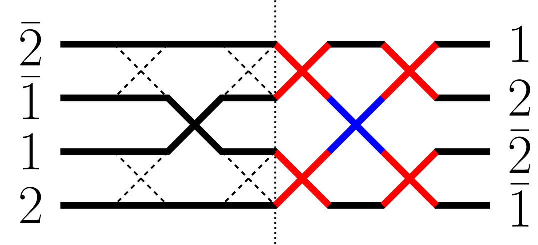
010 ↔  $\Big\}\ w = \bar{2}\bar{1}$

- A word is reduced if, in its wiring diagram, no pair of wires crosses more than once.

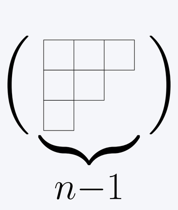
Involution words

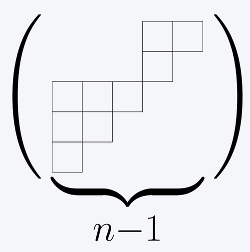
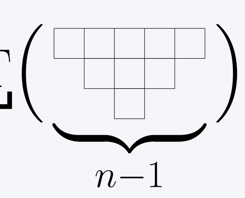
- Let $\mathcal{I}(B_n) := \{z \in B_n : z = z^{-1}\}$ be the set of *signed involutions*.
- The *Demazure product* $\circ$ is the associative binary operation $B_n \times B_n \to B_n$ with
  $$w \circ s_i = \begin{cases} ws_i & \text{if } \ell(ws_i) > \ell(w) \\ w & \text{if } \ell(ws_i) < \ell(w) \end{cases} \qquad \text{for } w \in B_n \text{ and } 0 \le i < n$$
- An *involution word* for $z \in \mathcal{I}(B_n)$ is a minimal-length word $\mathbf{a} = a_1 \cdots a_\ell$ with
  $$z = s_{a_\ell} \circ \cdots \circ s_{a_1} \circ 1 \circ s_{a_1} \circ \cdots \circ s_{a_\ell}.$$
  Considered by Richardson and Springer (1990) in their study of spherical varieties.
- Write $\hat{\mathcal{R}}(z)$ for the set of involution words for $z \in \mathcal{I}(B_n)$.
- Example: if $z = \bar{1}\bar{2} = (1, \bar{1})(2, \bar{2})$ then $\hat{\mathcal{R}}(z) = \{\mathbf{010}, \mathbf{101}\}$, as
  $$s_0 \circ s_1 \circ s_0 \circ 1 \circ s_0 \circ s_1 \circ s_0 = s_0 \circ s_1 \circ s_0 \circ s_1 \circ s_0 = s_0 s_1 s_0 s_1 = z.$$
- Involution words $\longleftrightarrow$ wiring diagrams with an approximate horizontal symmetry: a letter a_j corresponds to wires crossing at heights $\pm a_j$ and $\pm(a_j + 1)$, plus the same crossing reflected across $x = 0$ if this does not make two wires cross twice:

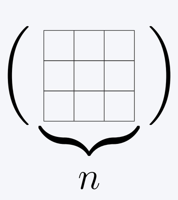
010 ↔ 

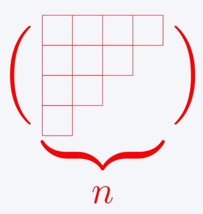
101 ↔ 

Enumeration formulas

Stanley (1984): $\#\mathcal{R}(w_n^A) = \#\mathrm{SYT}\Big($  $\Big)$ ($n-1$)

Hamaker, M., P. (2015): $\#\hat{\mathcal{R}}(w_n^A) = \#\mathrm{SYT}\Big($  $\Big)$ ($n-1$) $= \#\underset{\text{marked shifted}}{\underline{\mathrm{ShSYT}}}\Big($  $\Big)$ ($n-1$)

Haiman (1992): $\#\mathcal{R}(w_n^B) = \#\mathrm{SYT}\Big($  $\Big)$ (n)

M., P. (2018): $\#\hat{\mathcal{R}}(w_n^B) = \#\mathcal{R}(w_{n+1}^A) = \#\mathrm{SYT}\Big($  $\Big)$ (n)

Stanley symmetric functions

- The *(type C) Stanley symmetric function of* $w \in B_n$ *is*
  $$G_w := \sum_{\mathbf{a}=a_1a_2\cdots a_\ell \in \mathcal{R}(w)} \ \sum_{\substack{\mathbf{i}=(i_1 \le i_2 \le \cdots \le i_\ell) \\ i_{j-1} < i_{j+1} \text{ if } a_{j-1} < a_j > a_{j+1}}} 2^{\#\{i_1,\dots,i_\ell\}} x_{i_1} \cdots x_{i_\ell}.$$
  Outer sum is over reduced words; inner sum is over positive integer sequences.

Theorem [Worley (1984); Billey, Haiman (1995)]

It holds that $G_{w_n^B} = S_{(n^n)}$ where $S_\lambda := Q_{(\lambda+\delta)/\delta}$ for $\delta = (m, \dots, 2, 1)$ and $m \gg 0$.

- The *(type C) involution Stanley symmetric function* of $z \in \mathcal{I}(B_n)$ is
  $$\hat{G}_z := \sum_{\mathbf{a}=a_1a_2\cdots a_\ell \in \hat{\mathcal{R}}(z)} \ \sum_{\substack{\mathbf{i}=(i_1 \le i_2 \le \cdots \le i_\ell) \\ i_{j-1} < i_{j+1} \text{ if } a_{j-1} < a_j > a_{j+1}}} 2^{\#\{i_1,\dots,i_\ell\}} x_{i_1} \cdots x_{i_\ell},$$
  Equivalently, just replace $\mathcal{R}(w)$ by $\hat{\mathcal{R}}(z)$ in the definition of G_w.

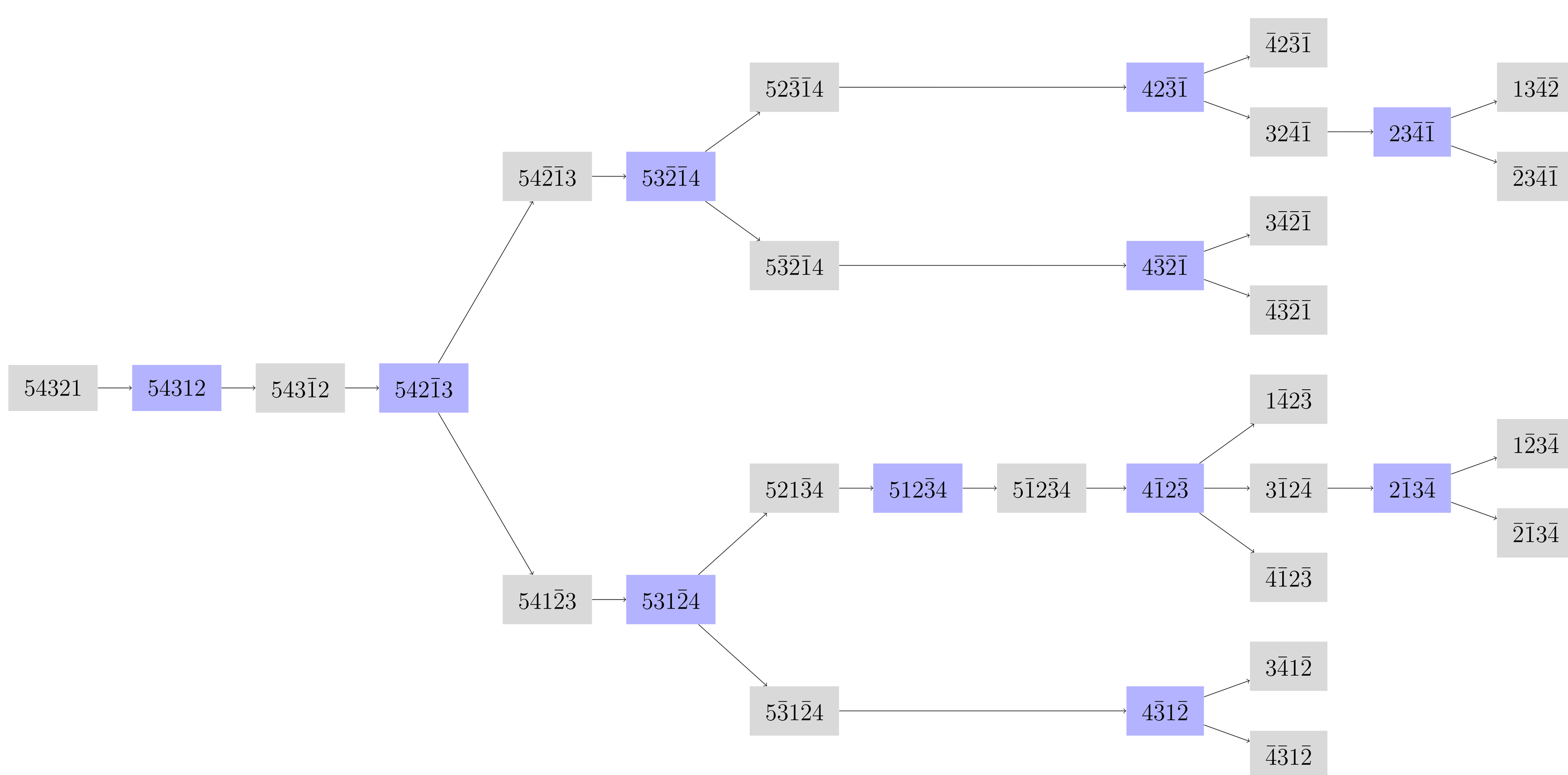
Theorem [M., P. (2018)]

It holds that $\hat{G}_{w_n^B} = G_{w_{n+1}^A} = S_{(n,n-1,\dots,2,1)}$ and $\#\hat{\mathcal{R}}(w_n^B) = \#\mathcal{R}(w_{n+1}^A)$

- The cardinalities $\#\mathcal{R}(w)$ and $\#\hat{\mathcal{R}}(z)$ can be read off as coefficients of G_w and $\hat{G}_z$.

Transitions

- For each $z \in \mathcal{I}(B_n)$, there is a set $\mathcal{A}(z) \subseteq B_n$ with
  $$\hat{\mathcal{R}}(z) = \bigsqcup_{w \in \mathcal{A}(z)} \mathcal{R}(w) \quad \text{and} \quad \hat{G}_z = \sum_{w \in \mathcal{A}(z)} G_w.$$
  For example, $\hat{G}_{\bar{1}\bar{2}} = G_{\bar{2}\bar{1}} + G_{1\bar{2}}$ as $\hat{\mathcal{R}}(\bar{1}\bar{2}) = \{\mathbf{010}, \mathbf{101}\} = \mathcal{R}(\bar{2}\bar{1}) \cup \mathcal{R}(1\bar{2})$.
- Given $v \in B_n$ and $r \in \mathbb{N}$, Billey's *transition formula* writes $\sum_u G_u = \sum_w G_w$ where u and w run over certain Bruhat covers of v depending on r.
- Proof idea: use transitions to gradually transform $G_{w_{n+1}^A}$ to $\sum_{w \in \mathcal{A}(w_n^B)} G_w = \hat{G}_{w_n^B}$.
- The directed bipartite graph below shows this strategy for $n = 4$. At each blue vertex v, the identity $\sum_{u \to v} G_u = \sum_{v \to w} G_w$ holds by the transition formula.



The unique source is 54321 and the sinks are exactly the elements of $\mathcal{A}(\bar{1}\bar{2}\bar{3}\bar{4})$.

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