Crystal structures for symmetric Grothendieck polynomials

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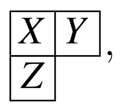
Background

The set of k-dimensional subspaces of $\mathbb{C}^n$ is called the Grassmannian. Modern Schubert calculus aims to study its geometry through its (connective) K-theory ring. The Schubert structure sheaf basis is represented by **symmetric Grothendieck polynomials** $\mathfrak{G}_\lambda(\mathbf{x};\beta)$, which form a basis for the ring of symmetric functions. A. Buch has shown that $\mathfrak{G}_\lambda(\mathbf{x};\beta)$ can be given as a sum over **set-valued tableaux** $\mathrm{SV}^n(\lambda)$, fillings of λ with nonempty subsets of $\{1,\dots,n\}$ such that rows are weakly increasing and columns strictly increasing in the following sense:



where $\max(X) \le \min(Y)$ and $\max(X) < \min(Z)$.

Grothendieck polynomials at $\beta = 0$ are precisely the Schur polynomials since β counts the **excess**, how far a set-valued tableaux is from being semistandard. Furthermore they are known to be Schur positive. Thus, a natural question is to determine a crystal structure, in the sense of Kashiwara, on set-valued tableaux. Since a product of Grothendieck polynomials is a finite positive sum of Grothendiecks, there should further be an extension of a crystal structure and the tensor product rule where $\mathrm{SV}^n(\lambda)$ become the irreducible objects that realizes the representation of some quantum group. One application of our proposed K-crystal structure is to study the more mysterious so-called **Lascoux polynomials**

$$L_{w\lambda}(\mathbf{x};\beta) := \varpi_{i_\ell} \cdots \varpi_{i_1} \mathbf{x}^\lambda, \qquad \text{where } \varpi_i = \frac{x_i(1+\beta x_{i+1})f - x_{i+1}(1+\beta x_i)s_i f}{x_i - x_{i+1}},$$

and $w = s_{i_1} \cdots s_{i_\ell}$ is some reduced expression. The Lascoux polynomials have no known geometric or representation theoretic interpretations, yet several conjectural combinatorial interpretations.

Goals:

- Construct a $U_q(\mathfrak{sl}_n)$-crystal structure on set-valued tableaux and relate this to other constructions.
- Construct a K-theoretic analog of crystals on set-valued tableaux.
- Determine a combinatorial interpretation of Lascoux polynomials.

Crystal structure

Let $T \in \mathrm{SV}^n(\lambda)$ and $i \in I := \{1,2,\dots,n-1\}$. Write $+$ above each column of T containing i but not $i+1$, and write $-$ above each column containing $i+1$ but not i. Cancel ordered pairs $-+$. If every $+$ (resp. $-$) thereby cancels, then $f_i(T) = 0$ (resp. $e_i(T) = 0$). Otherwise, let b be the box corresponding to the leftmost (resp. rightmost) uncanceled $+$ (resp. $-$). Otherwise f_iT (resp. e_iT) is given by replacing the i (resp. $i+1$) in b. If the result is not a set-valued tableau, then additionally move the i (resp. $i+1$) in the box immediately to the right (resp. left) into b.

Results

Theorem 1: *These crystal operators define a $U_q(\mathfrak{sl}_n)$-crystal structure on $\mathrm{SV}^n(\lambda)$ such that*

$$\mathrm{SV}^n(\lambda) \cong \bigoplus_{\mu \supseteq \lambda} B(\mu)^{\oplus M_\lambda^\mu},$$

where M_λ^μ denotes the number of $T \in \mathrm{SV}^n(\lambda)$ such that $e_iT = 0$ for all $i \in I$ (equivalently, the reading word, where we read the sets in decreasing order, is a Yamanouchi word).

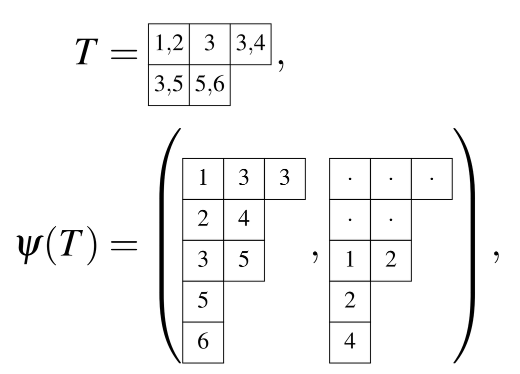
Corollary 2: *A. Buch's uncrowding bijection gives a crystal isomorphism such that connected component's are indexed by C. Lenart's flagged increasing tableaux.*

Theorem 3: *There exists a K-crystal structure on set-valued tableau $\mathrm{SV}^n(\lambda)$ when λ is a single row or a single column.*

Proposition 4: *There does* not *exist a (strong) K-crystal structure on $\mathrm{SV}^3(21)$.*

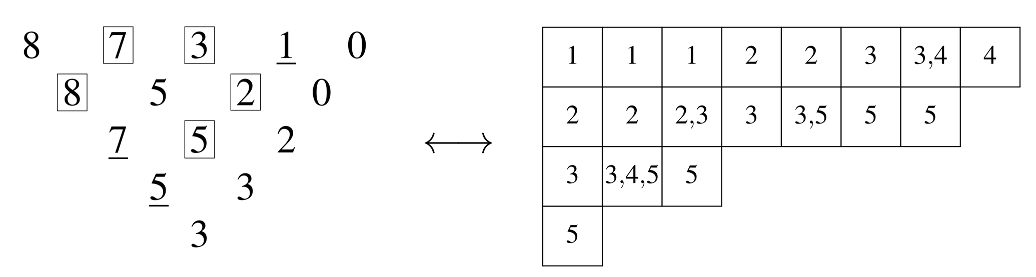
Uncrowding

The uncrowding bijection is given by RSK inserting each row of a set-valued tableaux into the previous result and recording the depths of the added boxes.



Marked GT patterns

A **Gelfand–Tsetlin (GT) pattern** is a sequence of partitions $(\lambda^{(j)})_{j=0}^n$ such that $\lambda^{(0)} = \emptyset$ and $\lambda^{(j)}/\lambda^{(j-1)}$ is a horizontal strip. By marking entries certain entries (i,j) provided $\lambda_{i+1}^{(j)} < \lambda_{i+1}^{(j-1)}$, we obtain a bijection with set-valued tableaux.



K-crystal structure

A **K-crystal structure** on $\mathrm{SV}^n(\lambda)$ is a $U_q(\mathfrak{sl}_n)$-crystal with **K-crystal operators** e_i^K, f_i^K such that

1. $\mathrm{SV}^n(\lambda)$ is connected with unique u_λ of weight λ such that $e_i u_\lambda = e_i^K u_\lambda = 0$ for all $i \in I$,
2. the **K-Demazure crystal**

$$\mathrm{SV}_w^n(\lambda) := \{ T \in \mathrm{SV}^n(\lambda) \mid (e_{i_\ell}^K)^{\max} e_{i_\ell}^{\max} \cdots (e_{i_1}^K)^{\max} e_{i_1}^{\max} T = u_\lambda \}$$

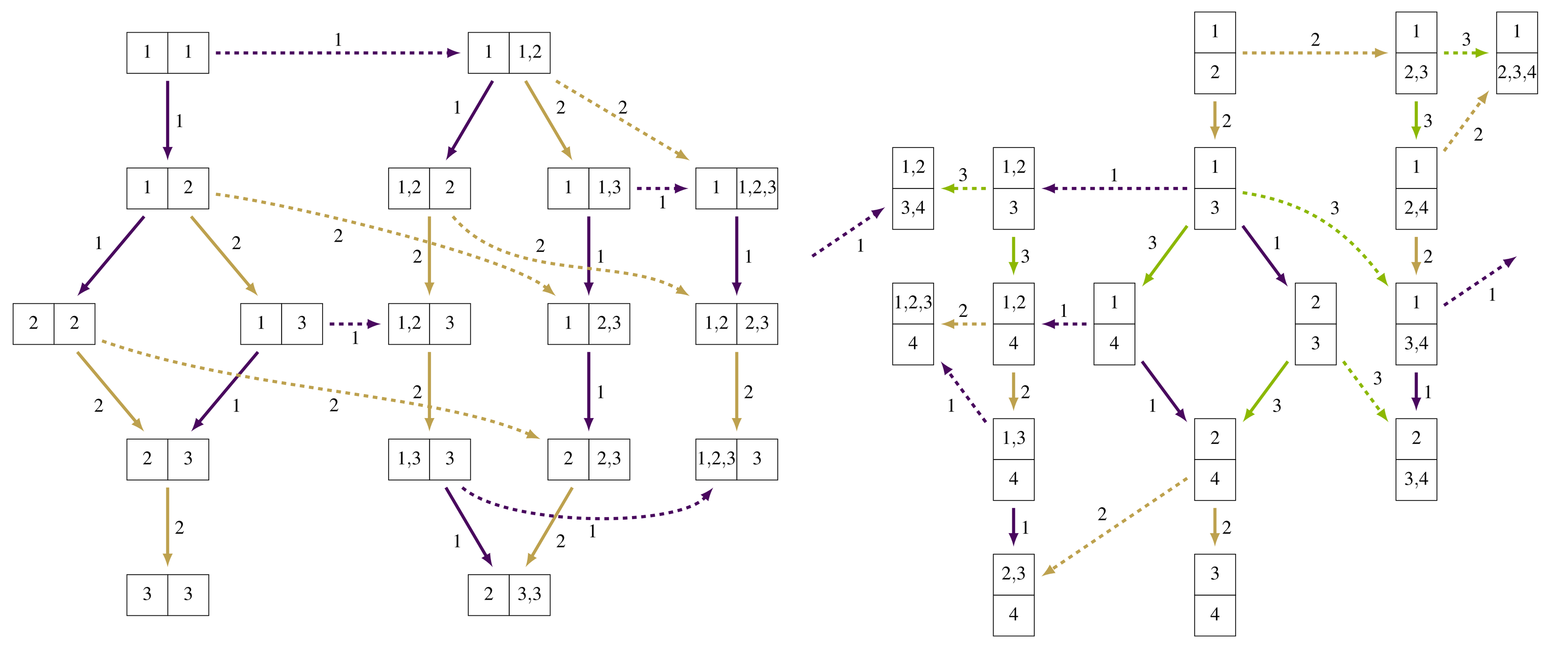
does not depend on the choice of reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, and

3. the character of $\mathrm{SV}_w^n(\lambda)$ is $L_{w\lambda}(\mathbf{x};\beta)$.

A **weak K-crystal** is when the K-Demazure crystal does depend on the choice of reduced expression and the character is only $L_{w\lambda}(\mathbf{x};\beta)$ for *some* reduced expression of w.

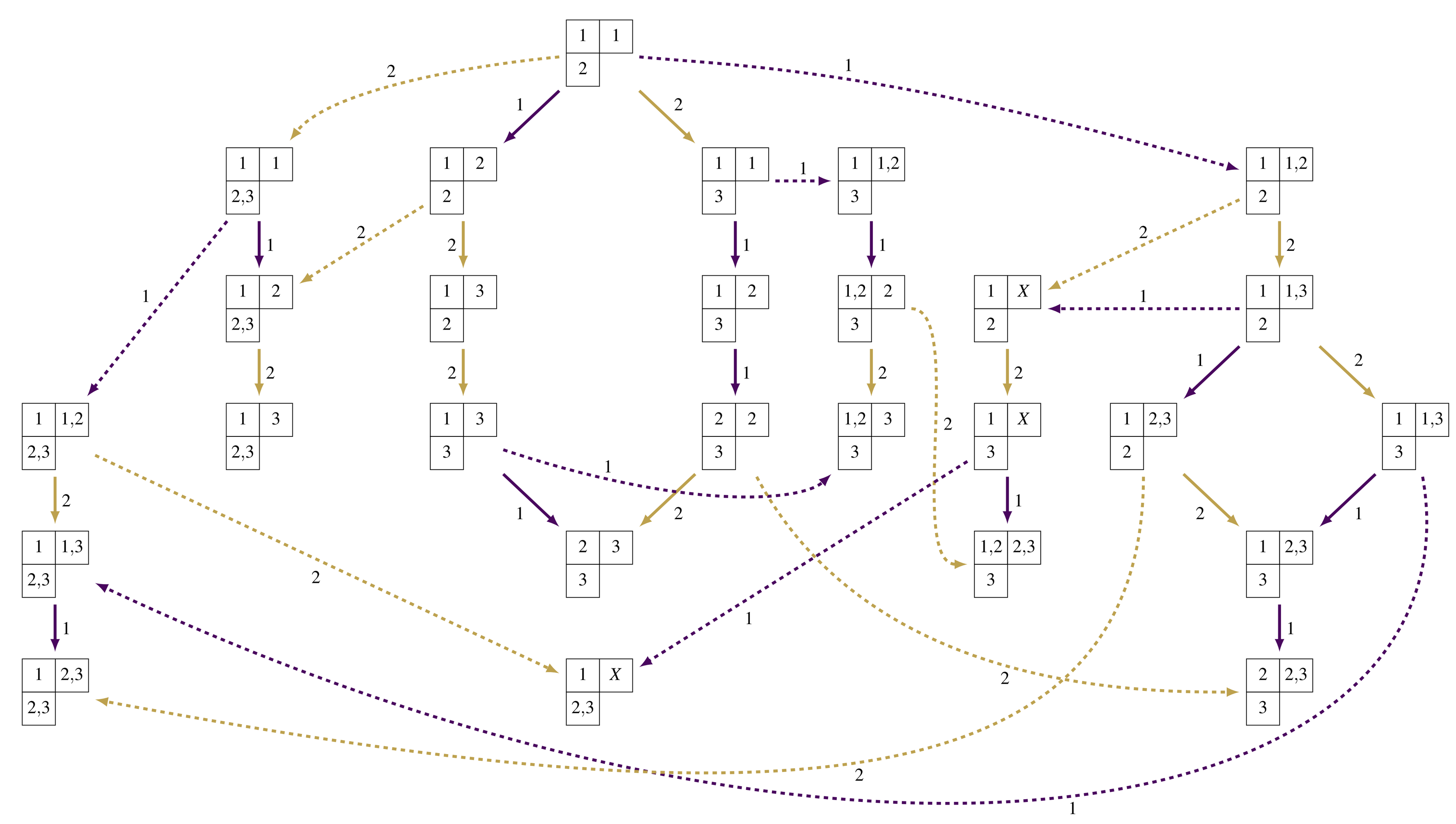
K-crystal examples

The K-crystal structures on $\mathrm{SV}^3(2)$ (left) and $\mathrm{SV}^4(11)$ (right):



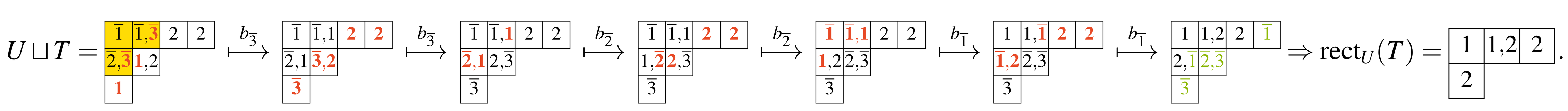
Weak K-crystal example

A weak K-crystal structure on $\mathrm{SV}^3(21)$ for the reduced word $w_0 = s_1 s_2 s_1$ (where $X = \{1,2,3\}$):



Rectification

Let U and T be set-valued tableau of shape λ and μ/λ respectively. We construct the tableau $U \sqcup T$ of shape λ by replacing every letter $k \mapsto \overline{k}$ in U and using the entries in T. Note $U \sqcup T$ is a set-valued tableau in the alphabet $\overline{1} < \overline{2} < \cdots < \overline{m} < 1 < 2 < \cdots < n$. We define operators b_i on set-valued tableaux whose entries are in an alphabet $\mathscr{A}$. Let j be the letter of $\mathscr{A}$ immediately greater than i. Let t_i be the **K-Bender–Knuth operator** t_i of T. Ikeda and T. Shimazaki given row-by-row interchanging i and j in each box such that there is not an j in the box below an i or an i in the box above an j and finally reversing all such boxes. Then, b_i acts by first applying t_i to the labels i and j, and then switching all instances of i and j. The **rectification** $\mathrm{rect}_U(T)$ of T (with respect to the rectification order U) is the semistandard tableau obtained by restricting $b_{\overline{1}}^{\circ n} \circ \cdots \circ b_{\overline{m-1}}^{\circ n} \circ b_{\overline{m}}^{\circ n}(U \sqcup T)$ to the unbarred alphabet. For an example, we have



We note that the rectification, in general, depends on the choice of U:

Open problems

- Construct a (weak) K-crystal structure on $\mathrm{SV}^n(\lambda)$.
- Determine a tensor product rule such that the result categorifies the K-theory ring of the Grassmannian.
- Determine a K-jeu de taquin rule such that K-rectification agrees with Buch's insertion algorithm.
- Determine a combinatorial rule that characterizes the set-valued tableaux that appear in $\mathrm{SV}_w^n(\lambda)$ for a given (weak) K-crystal structure.

References

[Buch02] A. S. Buch. A Littlewood–Richardson rule for the K-theory of Grassmannians. *Acta Math.*, 189(1):37–78, 2002.

[BS17] D. Bump and A. Schilling. *Crystal bases*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.

[IK14] T. Ikeda and T. Shimazaki. A proof of K-theoretic Littlewood–Richardson rules by Bender–Knuth-type involutions. *Math. Res. Lett.*, 21(2):333–339, 2014.

[Las01] A. Lascoux. Transition on Grothendieck polynomials. In *Physics and Combinatorics 2000 (Nagoya)*, pages 164–179 World Sci. Publ., River Edge, NJ, 2001.

[MPS18] C. Monical, O. Pechenik, and T. Scrimshaw. Crystal structures for symmetric Grothendieck polynomials. Preprint, arχiv:1807.03294.