Quarter-plane lattice paths with interacting boundaries: Kreweras and friends

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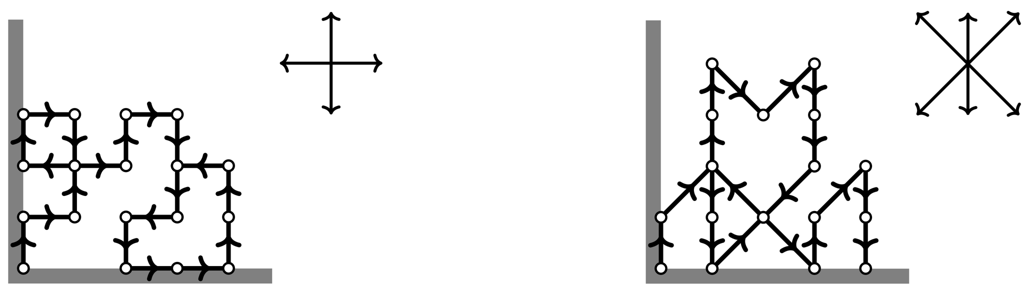
1. Introduction & background

Lattice paths in the quarter-plane have led to a rich body of research in combinatorics, with connections to probability theory, algebra, analysis and statistical physics. The standard approach is to choose a subset $\mathcal{S}$ of the 8 "short" steps $\{\rightarrow, \nearrow, \uparrow, \nwarrow, \leftarrow, \swarrow, \downarrow, \searrow\}$ and study the properties of random walks which are only allowed to take steps from $\mathcal{S}$. For example,



If $q_{n,i,j}$ is the number of quarter-plane paths of length n taking steps from $\mathcal{S}$ and ending at (i,j), a key object is the **generating function**

$$Q(t;x,y) = \sum_{n,i,j} q_{n,i,j} t^n x^i y^j. \tag{1}$$

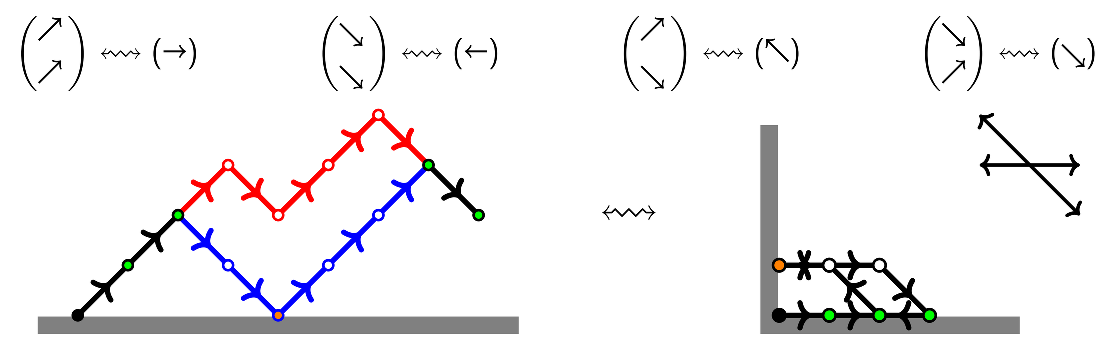
In particular, we are often interested in solving $Q(t;x,y)$ and determining whether it is **algebraic** or **D-finite** (or neither). The same questions can be asked about $Q(t;x,0)$, $Q(t;0,x)$ and $Q(t;0,0)$. See [2] for a seminal work in the field.

One connection with statistical physics was explored in [3]. The subject was a two-dimensional model of interacting directed polymers above an impenetrable surface. Two paths of length n start at the origin and take steps from $\{\nearrow, \searrow\}$; they can share vertices or edges but cannot cross; and they must remain on or above the x-axis. Two Boltzmann weights were introduced: weight a for contacts between the bottom walk and the wall, and weight c for each shared vertex except the origin. The goal was to compute the generating function

$$F(z;a,c) = \sum_{n,u,v} f_{n,u,v} z^n a^u c^v \tag{2}$$

where $f_{n,u,v}$ is the number of configurations of length n with u visits to the surface and v shared vertices. From a physics perspective, it is the singularity structure of $F(z;a,c)$ which is of interest, as this determines the **free energy** and the **phase diagram** of the model.

The connection with quarter-plane paths is a simple bijection. Every polymer configuration of size n can be mapped to a single quarter-plane path of length n taking steps in $\mathcal{S} = \{\leftarrow, \nearrow, \searrow\}$, via



Moreover, vertices in the boundary visited by the bottom polymer correspond to visits to the y-axis by the quarter-plane path, and shared vertices correspond to visits to the x-axis. So the numbers $f_{n,u,v}$ now count quarter-plane lattice paths of length n taking steps from $\mathcal{S}$, with u visits to the y-axis and v visits to the x-axis.

In [1] this idea of weighted boundaries was expanded to **all 23 non-isomorphic quarter-plane models with algebraic or D-finite generating functions**: a weight a was associated with the x-axis and weight b with the y-axis. For two models the generating function stays D-finite for all a,b. For some others a solution was found but the nature of the generating function depends on the values of a,b. For many others, however, no solution could be found at all.

Kreweras walks and **reverse Kreweras walks** are particularly interesting, as they are among the four algebraic models. In [1] it was shown that $Q(t;x,y)$ is still algebraic with boundary weights $a=b$ (note that $Q(t;0,0)$ is the same for both models), but when $a \neq b$ no solution was found. This is the problem that we address here.

2. Generating functions & functional equations

For given $\mathcal{S}$, let $q_{n,i,j,u,v,w}$ be the number of paths of length n which end at (i,j) with u visits to the positive x-axis, v visits to the positive y-axis and w visits to the origin (excluding the first vertex). Then (1) can be enhanced:

$$Q(t;x,y;a,b,c) = \sum_{n,i,j,u,v,w} q_{n,i,j,u,v,w} t^n x^i y^j a^u b^v c^w. \tag{3}$$

For brevity we will usually just write $Q(x,y)$. In addition, we use the shorthand $\bar{x} = \frac{1}{x}$ and $\bar{y} = \frac{1}{y}$, and similar notation for a,b,c.

First define

$$S(x,y) = \sum_{(i,j)\in\mathcal{S}} x^i y^j. \tag{4}$$

Also define $A_-(x) = [\bar{y}]S(x,y)$, $B_-(y) = [\bar{x}]S(x,y)$ and $C_- = [\bar{x}\bar{y}]S(x,y)$. So for example, for Kreweras walks,

$$S(x,y) = xy + \bar{x} + \bar{y}, \qquad A_-(x) = B_-(y) = 1, \qquad C_- = 0. \tag{5}$$

A walk of length n is either empty (if $n=0$) or can be constructed by taking a walk of length $n-1$ and appending a step from $\mathcal{S}$. This can be written as $Q(x,y) = 1 + tS(x,y)Q(x,y)$. However, there are two things we need to take into account:

- We must not allow walks to cross the x- or y-axis.
- When a walk steps onto the x-axis, the y-axis, or the origin, it must accrue a weight a, b or c respectively.

The first problem is dealt with by the now **well-known functional equation**

$$Q(x,y) = 1 + tS(x,y)Q(x,y) - tA_-(x)\bar{y}Q(x,0) - tB_-(y)\bar{x}Q(0,y) + tC_-\bar{x}\bar{y}Q(0,0). \tag{6}$$

But this does not accommodate the extra boundary weights. To do so, we must also consider walks ending one step away from the axes (since they can then step onto the axes). These can then be written in terms of $Q(x,0)$, $Q(0,y)$ and $Q(0,0)$. This all takes some work, but the final result is remarkably simple.

Theorem 1

Let $K(t;x,y) \equiv K(x,y) = 1 - tS(x,y)$. Then with all three boundary weights a,b,c,

$$\begin{aligned} K(x,y)Q(x,y) = \bar{c} &+ (1 - \bar{a} - tA_-(x)\bar{y})Q(x,0) + (1 - \bar{b} - tB_-(y)\bar{x})Q(0,y) \\ &+ ((\bar{a} + \bar{b} - \bar{c} - 1) + tC_-\bar{x}\bar{y})Q(0,0). \end{aligned} \tag{7}$$

For Kreweras walks this functional equation reads

$$(1 - t(xy + \bar{x} + \bar{y}))Q(x,y) = \bar{c} + (1 - \bar{a} - t\bar{y})Q(x,0) + (1 - \bar{b} - t\bar{x})Q(0,y) + (\bar{a} + \bar{b} - \bar{c} - 1)Q(0,0) \tag{8}$$

and for reverse Kreweras walks it is

$$(1 - t(x + y + \bar{x}\bar{y}))Q(x,y) = \bar{c} + (1 - \bar{a} - t\bar{x}\bar{y})Q(x,0) + (1 - \bar{b} - t\bar{x}\bar{y})Q(0,y) + ((\bar{a} + \bar{b} - \bar{c} - 1) + t\bar{x}\bar{y})Q(0,0). \tag{9}$$

3. Main results

Our main results are the following two theorems.

Theorem 2: Reverse Kreweras walks

For all a,b,c the generating functions $Q(x,y)$, $Q(x,0)$, $Q(0,y)$ and $Q(0,0)$ are algebraic.

Theorem 3: Kreweras walks

When $a \neq b$ the generating function $Q(0,0)$ is algebraic (being the same as reverse Kreweras), while $Q(x,y)$, $Q(x,0)$ and $Q(0,y)$ are D-finite.

4. Useful tools

Here we outline a set of tools which have been developed over the years to solve quarter-plane walk models, all of which will be used to solve the Kreweras and reverse Kreweras models. When these tools are brought together to solve a system of functional equations, the method is known as the **algebraic kernel method**. See [2] for more details.

1 The group of $\mathcal{S}$: This is a group $\mathcal{G}$ of birational transformations of (x,y) which leave $S(x,y)$ (or equivalently, $K(x,y)$) unchanged. If we let $A_+(x) = [y]S(x,y)$ and $B_+(y) = [x]S(x,y)$, then it is generated by the two transformations

$$\Phi : (x,y) \mapsto \left(\bar{x}\frac{B_-(y)}{B_+(y)}, y\right) \qquad \text{and} \qquad \Psi : (x,y) \mapsto \left(x, \bar{y}\frac{A_-(x)}{A_+(x)}\right) \tag{10}$$

For all algebraic and D-finite quarter-plane models, $\mathcal{G}$ is finite.

2 The (full) orbit sum: For each $g \in \mathcal{G}$, apply g to the functional equation (6) (without boundary weights) or (7) (with boundary weights) to obtain $|\mathcal{G}|$ equations. Then take a linear combination to eliminate all $Q(\cdot,0)$ and $Q(0,\cdot)$ terms.

3 The half orbit sum: As the name suggests, we take only half of the equations from the full orbit sum and select certain terms to be eliminated – usually those $Q(\cdot,0)$ or $Q(0,\cdot)$ terms with negative powers of x or y as arguments.

4 Coefficient extraction: After eliminating terms, one typically rearranges the resulting equation somewhat and then extracts either a fixed power $[x^i]$ or $[y^j]$, or a generalised coefficient like all the positive powers of x, which we write as $[x^>]$. This may be performed multiple times. Two further techniques are sometimes required.

5 Partial fraction expansion of $\frac{1}{K(x,y)}$: In order to perform a coefficient extraction, one sometimes needs to deal with a rational function with $K(x,y)$ in the denominator. This is non-trivial, as $1/K(x,y)$, viewed as a power series in t, has infinitely many positive and negative powers of x and y.

Suppose we wish to extract $[y^j]\frac{1}{K(x,y)}$. Set $A_0(x) = [y^0]S(x,y)$, and then $\Delta(t;x) \equiv \Delta(x) = (1 - tA_0(x))^2 - 4t^2A_-(x)A_+(x)$ is the discriminant of $K(x,y)$. The two roots of the kernel $K(x,y)$, viewed as a Laurent polynomial in y, are

$$Y_\pm(t;x) \equiv Y_\pm(x) = \frac{1 - tA_0(x) \pm \sqrt{\Delta(x)}}{2tA_+(x)}. \tag{11}$$

The coefficient of y^j can then be easily obtained from the partial fraction expansion of $\frac{1}{K(x,y)}$,

$$\frac{1}{K(x,y)} = \frac{1}{\sqrt{\Delta(x)}}\left(-1 + \frac{1}{1 - \bar{y}Y_-(x)} + \frac{1}{1 - y/Y_+(x)}\right). \tag{12}$$

6 Canonical factorisation of $\Delta(x)$: The discriminant $\Delta(x)$ is a Laurent polynomial in x, with valuation δ and degree d (say). Then it has $\delta + d$ roots $X_i(t) \equiv X_i$ which are Puiseux series in t. Of these, $X_1, \ldots, X_\delta$ vanish as $t \to 0$ while $X_{\delta+1}, \ldots, X_{\delta+d}$ diverge. Writing

$$\Delta_-(t;x) \equiv \Delta_-(x) = \prod_{i=1}^{\delta}(1 - \bar{x}X_i), \qquad \Delta_+(t;x) \equiv \Delta_+(x) = \prod_{i=\delta+1}^{\delta+d}(1 - x/X_i), \qquad \Delta_0(t) \equiv \Delta_0 = (-1)^d[x^d]\Delta(x)\prod_{i=\delta+1}^{\delta+d} X_i, \tag{13}$$

we have $\Delta(x) = \Delta_-(x)\Delta_+(x)\Delta_0$. Moreover, these three factors are power series in t, with constant term 1 and coefficients in $\mathbb{C}$, $\mathbb{C}[\bar{x}]$ and $\mathbb{C}[x]$ respectively.

7 Cancelling a kernel: In many situations (not just quarter-plane walks) may obtain an equation of the form

$$P(t;x)G(t;x) = C_0(t;x) + C_1(t;x)G_1(t) + \cdots + C_k(t;x)G_k(t) \tag{14}$$

where $P(t;x)$ is a polynomial in x and the G terms are power series in t. If P has k different roots $X_1, \ldots, X_k$ which may be validly substituted into $G(t;x)$ (typically this means each X_i is a power series in t) then we obtain k linear equations for the unknowns $G_1, \ldots, G_k$. With a bit of luck these equations will be linearly independent. This can be generalised to multiple equations or variables.

5. Outline of solution

For comparison, unweighted Kreweras walks can be solved as follows. We use the notation $Q_{ij}(t) \equiv Q_{ij} = [x^iy^h]Q(x,y)$ and $Q^{\mathrm{d}}_h(t;z) \equiv Q^{\mathrm{d}}_h(z) = \sum_i z^i[x^iy^{i+h}]Q(x,y)$ (commonly called the **diagonal** of the generating function).

- [1] $\mathcal{G} \cong D_3$, generated by $\Phi : (x,y) \mapsto (\overline{xy}, y)$ and $\Psi : (x,y) \mapsto (x, \overline{xy})$.
- [3] Use (x,y), $(\overline{xy}, y)$ and $(\overline{xy}, x)$; eliminate $Q(\overline{xy},0)$ and $Q(0,y)$; use $Q(0,x) = Q(x,0)$; this gives an equation in $Q(x,y)$, $Q(\overline{xy},y)$, $Q(\overline{xy},x)$, $Q(x,0)$, $Q(0,0)$ with rational coefficients.
- [5] Divide by $K(x,y)$ and expand; now some coefficients are algebraic, with $\sqrt{\Delta(x)}$ in the denominator.
- [4] Extract $[y^0]$, giving an equation in $Q(x,0)$ and $Q^{\mathrm{d}}_0(\bar{x})$; coefficients still algebraic.
- [6] Factorise $\Delta(x)$ and separate positive and negative powers of x.
- [4] Separately extract $[x^>]$ to get an expression for $Q(x,0)$ and $[x^\leq]$ to get an expression for $Q^{\mathrm{d}}_0(\bar{x})$; back-substitute to obtain all other generating functions; everything still algebraic.

For weighted reverse Kreweras walks, our approach is more involved:

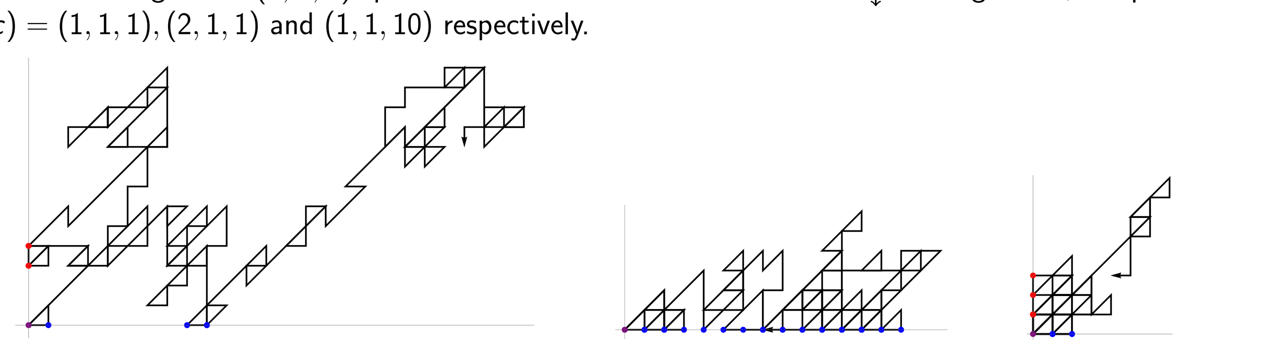
- [1] $\mathcal{G}$ is the same as above.
- [2] The RHS entirely cancels; left with $Q(x,y)$, $Q(\overline{xy},y)$, $Q(y,\overline{xy})$, $Q(y,x)$, $Q(\overline{xy},x)$, $Q(x,\overline{xy})$; all coefficients rational.
- [4] Extract $[y^0]$, giving an equation with $Q(x,0)$, $Q(0,x)$, $Q^{\mathrm{d}}_0(\bar{x})$, $Q^{\mathrm{d}}_1(\bar{x})$, $Q(0,0)$; all coefficients Laurent polynomials in x.
- [4] Separately extract $[x^\geq]$ and $[x^<]$; the former leads to an equation with $Q(x,0)$, $Q(0,x)$, $Q(0,0)$, and the latter $Q^{\mathrm{d}}_0(\bar{x})$, $Q^{\mathrm{d}}_1(\bar{x})$, $Q(0,0)$; all coefficients still Laurent polynomials in x.
  - [3] As in the unweighted case, use (x,y), $(\overline{xy},y)$ and $(\overline{xy},x)$; eliminate $Q(\overline{xy},0)$ and $Q(0,y)$; left with $Q(x,y)$, $Q(\overline{xy},y)$, $Q(\overline{xy},x)$, $Q(x,0)$, $Q(0,x)$, $Q(0,0)$ with rational coefficients.
  - [5] Divide by $K(x,y)$ and expand; now some coefficients are algebraic, with $\sqrt{\Delta(x)}$ in the denominator.
  - [4] Extract $[y^0]$, giving an equation in $Q(x,0)$, $Q(0,x)$, $Q^{\mathrm{d}}_0(\bar{x})$, $Q^{\mathrm{d}}_1(\bar{x})$, $Q(0,0)$; coefficients algebraic.
- Eliminate $Q(0,x)$ and $Q^{\mathrm{d}}_1(\bar{x})$; coefficients still algebraic.
- [4] Factorise $\Delta(x)$ and separate positive and negative powers of x.
- [4] Separately extract $[x^>]$ and $[x^\leq]$ (and switch $x \mapsto \bar{x}$ in the second), giving two equations of the form (14):
  $$S(x)Q(x,0) = \sigma(x) + \sigma_{0,0}(x)Q(0,0) + \sigma_{0,1}(x)Q_{0,1} + \sigma_{1,0}(x)Q_{1,0}$$
  $$T(x)Q^{\mathrm{d}}_0(x) = \tau(x) + \tau_{0,0}(x)Q(0,0) + \tau_{0,1}(x)Q_{0,1} + \tau_{1,0}(x)Q_{1,0}$$
  where S and T are each products of two quadratics, and the σ and τ coefficients are algebraic.
- [7] Separately cancel S and T to get four linear equations, of which three are independent; get algebraic solutions to $Q(0,0)$, $Q_{0,1}$ and $Q_{1,0}$; back-substitute to obtain solutions to everything else.

For Kreweras walks, things are similar but messier – the RHS of the full orbit sum does not cancel, so we are forced to divide by $K(x,y)$ before extracting $[y^0]$. The result has algebraic coefficients, and then when extracting $[x^\geq]$ some coefficients become **D-finite**. Moreover, at the end, we have too few independent equations, and must "import" $Q(0,0)$ from the reverse Kreweras model.

6. Further questions

There are still many models which have not been solved with general a,b,c weights. Perhaps the most tempting are the other two models which are algebraic without weights – **Gessel walks** and **double Kreweras walks**.

The **asymptotic behaviour** of the coefficients and the corresponding **phase diagrams** are also not yet understood. We expect **phase transitions** to occur in certain regions of (a,b,c)-space. Below are shown Kreweras walks of length 200, sampled from the Boltzmann distribution at $(a,b,c) = (1,1,1), (2,1,1)$ and $(1,1,10)$ respectively.



7. References

[1] N.R. Beaton, A.L. Owczarek and A. Rechnitzer. "Exact solution of some quarter plane walks with interacting boundaries". To appear in *Electron. J. Combin.* arXiv:1807.08853.

[2] M. Bousquet-Mélou and M. Mishna. "Walks with small steps in the quarter plane". *Algorithmic Probability and Combinatorics*, Contemporary Mathematics 520. AMS, 2010.

[3] R. Tabbara, A.L. Owczarek and A. Rechnitzer. "An exact solution of two friendly interacting directed walks near a sticky wall". *J. Phys. A: Math. Theor.* **47** (2014), 015202.

For further details on this work see

[4] N.R. Beaton, A.L Owczarek and R. Xu. "Quarter-plane lattice paths with interacting boundaries: the Kreweras and reverse Kreweras models". arXiv:1905.10908.

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