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Categorifying Combinatorial Hopf algebras

$\mathsf{NCSym} = \bigoplus_{n\geq 0} \mathsf{NCSym}_n$ — Symmetric functions in non-commuting variables

$\mathsf{NCSym}_n = \mathbb{C}\text{-span}\{m_\lambda \mid \lambda \text{ a set partition of } \{1,2,\ldots,n\}\}$

[Novelli—Thibon]

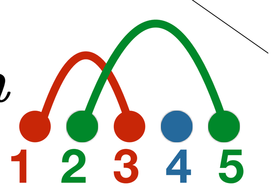
CQSym

Thm (AIMS—BEST—NO—END—GRINNING—NINNIES)

Thm (A—T)

$\mathrm{rep}(\mathsf{UT};\mathsf{AG}) \cap \mathrm{rep}(\mathsf{U};\mathsf{NP})$

rep(UT;AG)

m  $= X_1X_2X_1X_3X_2 + \cdots$

$\mathrm{nn}(\lambda) =$ keep only the smallest arc in nested arcs of λ

$\sum_{\mathrm{nn}(\lambda)=\mu} m_\lambda \quad \delta_\mu$

$\frac{|\mathsf{UT}_n|}{|\mathsf{U}_\mu|} p_\mu \quad \mathrm{Ind}_{\mathsf{U}_\mu}^{\mathsf{UT}_n}(\mathbb{1})$

$p_\mu = \sum_{\lambda\succeq\mu} m_\lambda$

new basis? $\quad \chi^\mu$

δ_λ

we have explicit formulas for the Hopf structure of these bases

no symmetric function interpretation ?

superclasses indexed by certain pairs of posets (where one gives a normal subgroup of the other)

we classify the primitives and give structure constants for the canonical bases $\longrightarrow$ rep(U;NP)

Representation theory of unipotent groups

A supercharacter theory of a group G is a set of characters Ch that are
- orthogonal (w.r.t. the usual inner product)
- the vector space spanned by Ch
  - contains regular character
  - forms an algebra under point-wise multiplication.

Remark. Under these conditions there is a unique partition Cl of G (whose blocks are called superclasses) satisfying

$$\mathrm{rep}(G;\mathsf{Cl}) := \{\psi : G \to \mathbb{C} \mid \psi \text{ constant on the blocks of Cl}\} = \mathbb{C}\text{-span}\{\mathsf{Ch}\}.$$

Normal lattice supercharacter theory. Given a sublattice L of the lattice of normal subgroups of G, we obtain a supercharacter theory such that

$$\mathrm{rep}(G;\mathsf{L}) := \mathbb{C}\text{-span}\{\mathsf{Ch}\} = \mathbb{C}\text{-span}\{\mathrm{Ind}_N^G(\mathbb{1}) \mid N \in \mathsf{L}\}.$$

Remark. In fact, the identities

$$\mathrm{Ind}_N^G(\mathbb{1}) = \sum_{O\supseteq N} \chi^O = \frac{|G|}{|N|}\sum_{M\subseteq N}\delta_M$$

relate the three canonical bases

(1) $\delta_N(g) = \begin{cases} 1 & \text{if } g\in N, g\notin M\subsetneq N,\\ 0 & \text{otherwise,}\end{cases}$

(2) $\mathrm{Ind}_N^G(\mathbb{1})$,

(3) $\chi^N \in \mathsf{Ch}$,

for $N \in \mathsf{L}$.

Pattern groups

Given a poset $\mathcal{P}$, define a group

$$\mathsf{U}_\mathcal{P} = \{f : \{i \prec_\mathcal{P} j\} \to \mathbb{F}_q\}$$

with

$$(f\circ g)(i,k) = \sum_{i\preceq j\preceq k} f(i,j)g(j,k).$$

Example.

$$\mathsf{U}_{1\prec 2\prec\cdots\prec n} \cong \mathsf{UT}_n = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1\end{bmatrix} \subseteq \mathsf{GL}_n(\mathbb{F}_q)$$

A lattice for pattern subgroups. Given a poset $\mathcal{R}$, let

$$\mathsf{NP} = \{\mathsf{U}_\mathcal{P} \lhd \mathsf{U}_\mathcal{R}\}.$$

Let

$$\mathrm{rep}(\mathsf{U};\mathsf{NP}) = \bigoplus_{n\geq 0}\ \bigoplus_{\mathcal{R} \text{ poset of } \{1,2,\ldots,n\}} \mathrm{rep}(\mathsf{U}_\mathcal{R};\mathsf{NP})$$

with product

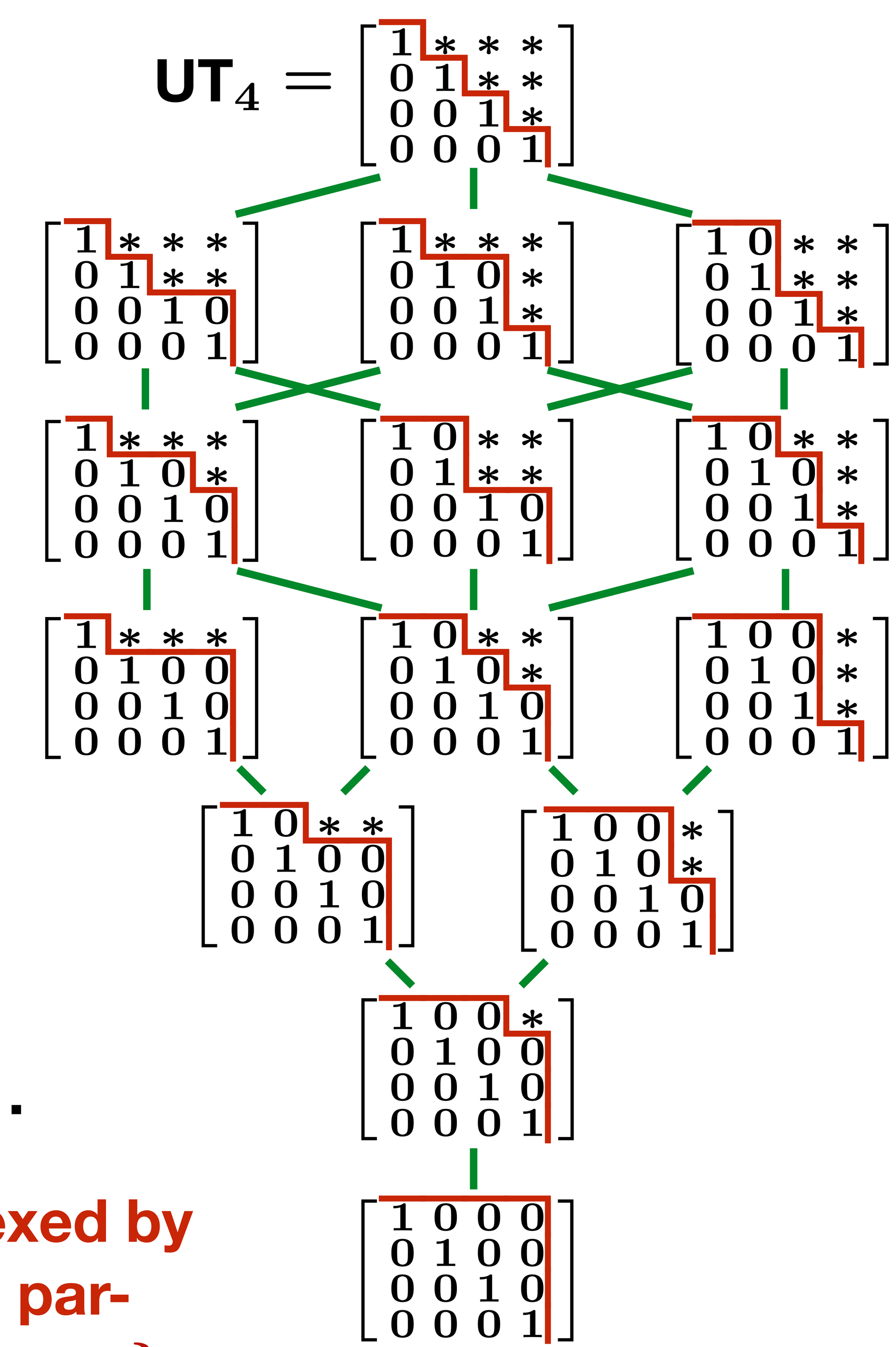
$$\psi\cdot\theta = \mathrm{Inf}_{\mathsf{U}_\mathcal{R}\times\mathsf{U}_\mathcal{Q}}^{\mathsf{U}_{\mathcal{R}\cdot\mathcal{Q}}}(\psi\otimes\theta)$$

and co-product

$$\Delta(\psi) = \sum_{A\subseteq\mathcal{R}} \mathrm{Res}_{\mathsf{U}_{\mathcal{R}_A}\times\mathsf{U}_{\mathcal{R}_{\bar{A}}}}^{\mathsf{U}_\mathcal{R}}(\psi)$$

$\psi \in \mathrm{rep}(\mathsf{U}_\mathcal{R};\mathsf{NP})$, $\theta \in \mathrm{rep}(\mathsf{U}_\mathcal{Q};\mathsf{NP})$

$\mathsf{UT}_4 =$



Remarks. Andrews first defines this Hopf structure, which may also be realized as a Hopf monoid. However, Andrews did not have the general lattice theory at his disposal and used only two canonical bases. The Hopf algebra

$$\mathrm{rep}(\mathsf{UT};\mathsf{AG}) = \bigoplus_{n\geq 0}\mathrm{rep}(\mathsf{UT}_n;\mathsf{AG}) \text{ with } \mathrm{rep}(\mathsf{UT};\mathsf{AG})\cap\mathrm{rep}(\mathsf{U};\mathsf{NP}) = \bigoplus_{n\geq 0}\mathrm{rep}(\mathsf{UT}_n;\mathsf{NP}).$$

↑ superclasses indexed by set partitions of $\{1,2,\ldots,n\}$

↑ superclasses indexed by non-nesting set partitions of $\{1,2,\ldots,n\}$

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