Cubic realization of Tamari interval lattices

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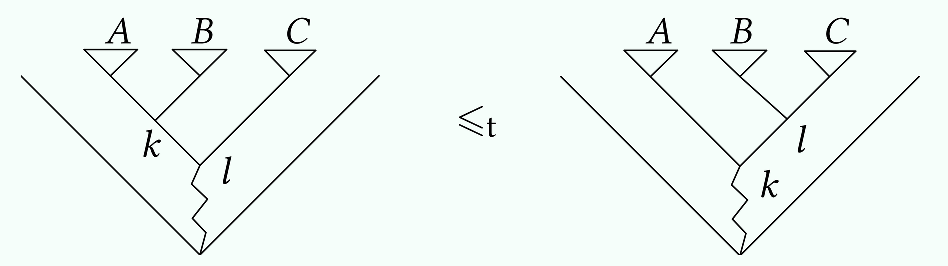
Tamari interval lattices

Tamari posets [Tamari, 1962] are defined on the set of binary trees with n nodes, endowed with the partial relation $\leqslant_t$, given by the reflexive and transitive closure of the right rotation:



Tamari interval posets are defined on the set of pairs of binary trees $[S, T]$ such that $S \leqslant_t T$, endowed with the partial order $\leqslant_{ti}$:

$$[S, T] \leqslant_{ti} [S', T'] \quad \text{if and only if} \quad S \leqslant_t S' \text{ and } T \leqslant_t T'.$$

We denote $\mathcal{TI}_n$ the set of Tamari intervals of size n.

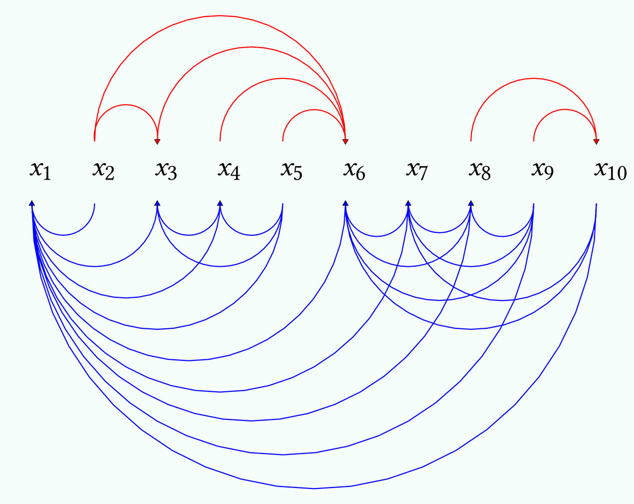
An interval-poset P of size n is a partial order $\lhd$ on the set $\{x_1, \dots, x_n\}$ such that, for any $i < k$,

- if $x_k \lhd x_i$ then for all x_j such that $i < j < k$, one has $x_j \lhd x_i$,
- if $x_i \lhd x_k$ then for all x_j such that $i < j < k$, one has $x_j \lhd x_k$.

We denote $\mathcal{IP}_n$ the set of interval-posets of size n.

There is a bijection $\rho : \mathcal{IP}_n \to \mathcal{TI}_n$ [Châtel, Pons, 2015].

We draw interval-posets as oriented graphs with decreasing relations in blue and increasing relations in red.



Example

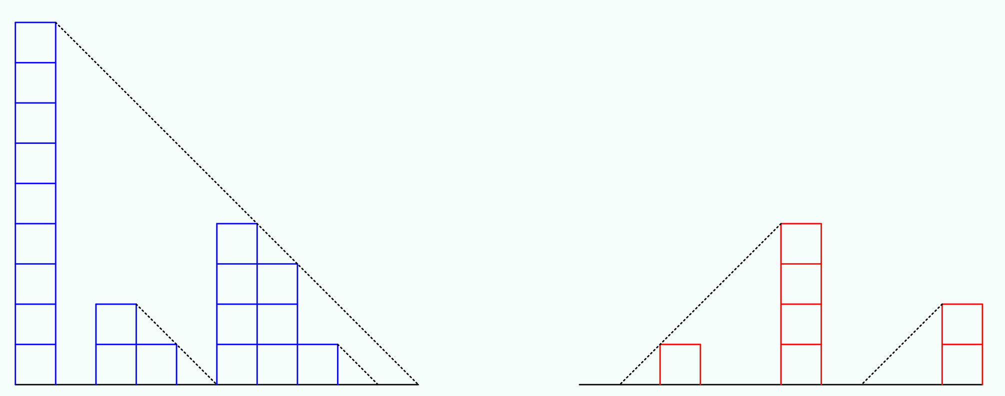
Interval-poset of size 10.

Tamari interval diagrams

A Tamari diagram is a word $u = u_1 \dots u_n$ of integers such that

- $0 \leqslant u_i \leqslant n - i$ for all $i \in [n] = \{1, \dots, n\}$;
- $u_{i+j} \leqslant u_i - j$ for all $i \in [n]$ and $0 \leqslant j \leqslant u_i$.

Symmetrically, a dual Tamari diagram is a word $v = v_1 \dots v_n$ such that the mirror image of v is a Tamari diagram.



$u = 9021043100$ $\qquad$ $v = 0010040002$

Let u (resp. v) be a (resp. dual) Tamari diagram of size n. The diagrams u and v are compatible if for all $1 \leqslant i < j \leqslant n$ such that $j - i \leqslant u_i$, we have $v_j < j - i$.

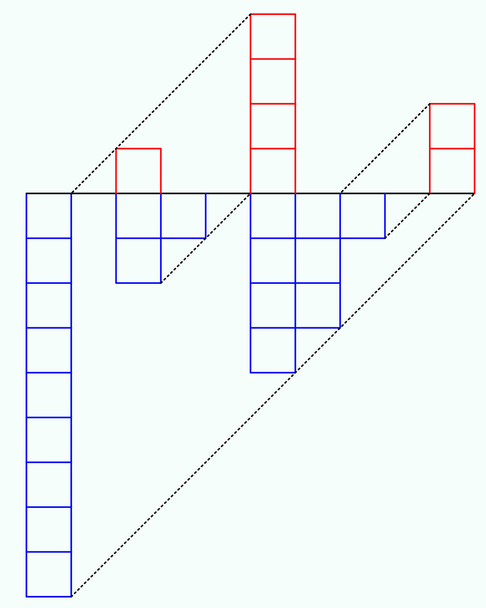
If u and v are compatible, then the pair (u, v) is a Tamari interval diagram.

We denote $\mathcal{TID}_n$ the set of Tamari interval diagrams of size n.

Let χ be the map sending a Tamari interval diagram (u, v) of size n to the binary relation $\lhd$ on $\{x_1, \dots, x_n\}$ where for all $i \in [n]$ and $0 \leqslant l \leqslant u_i$, $x_{i+l} \lhd x_i$, and for all $i \in [n]$ and $0 \leqslant k \leqslant v_i$, $x_{i-k} \lhd x_i$.

Theorem [C., 2019]

The map χ is a bijection from $\mathcal{TID}_n$ to $\mathcal{IP}_n$.



The Tamari diagram u is drawn in blue and the dual Tamari diagram v is drawn in red.

Example

Tamari interval diagram $(9021043100, 0010040002)$.

Cubic coordinates

Let c be a $(n-1)$-tuple with entries in $\mathbb{Z}$. We say that c is a cubic coordinate if the pair (u, v), where u is the word defined by $u_n = 0$ and for all $i \in [n-1]$ by

$$u_i = \max(c_i, 0),$$

and v is the word defined by $v_1 = 0$ and for all $2 \leqslant i \leqslant n$ by

$$v_i = |\min(c_{i-1}, 0)|,$$

is a Tamari interval diagram. The size of a cubic coordinate is its number of entries plus one. The set of cubic coordinates of size n is denoted by $\mathcal{CC}_n$.

Example

$$v = 0\,0\,1\,0\,0\,4\,0\,0\,0\,2 \qquad u_i - v_{i+1}$$
$$u = 9\,0\,2\,1\,0\,4\,3\,1\,0\,0 \qquad \longrightarrow \qquad (9, -1, 2, 1, -4, 4, 3, 1, -2).$$

There is a bijection $\phi : \mathcal{CC}_n \to \mathcal{TID}_n$.

Some properties of Tamari intervals translate nicely on cubic coordinates or on Tamari interval diagrams, for instance:

- A cubic coordinate c of size n is synchronized if for all $i \in [n-1]$, $c_i \neq 0$. The set of synchronized cubic coordinates of size n is denoted by $\mathcal{CC}_n^{sync}$. (synchronized Tamari intervals, [Préville-Ratelle, Viennot, 2017])
- A Tamari interval diagram (u, v) of size n is new if the following conditions are satisfied:
  - $0 \leqslant u_i \leqslant n - i - 1$ for all $i \in [n-1]$;
  - $0 \leqslant v_j \leqslant j - 2$ for all $j \in \{2, \dots, n\}$;
  - $u_k < l - k - 1$ or $v_l < l - k - 1$ for all $k, l \in [n]$ such that $k + 1 < l$.

  (new Tamari intervals, [Chapoton, 2006])
- If (u, v) is synchronized then (u, v) is not new.

Let $c, c' \in \mathcal{CC}_n$.

Partial order: $c \leqslant_{cc} c'$ if and only if $c_i \leqslant c'_i$ for all $i \in [n-1]$.

Covering relation: $c \lessdot c'$ if and only if there is exactly one $i \in [n-1]$ such that $c_i < c'_i$, and if there is a $c'' \in \mathcal{CC}_n$ such that $c \leqslant_{cc} c'' \leqslant_{cc} c'$, then either $c = c''$ or $c' = c''$.

Let $\psi = \phi^{-1} \circ \chi^{-1} \circ \rho^{-1}$.

Theorem [C., 2019]

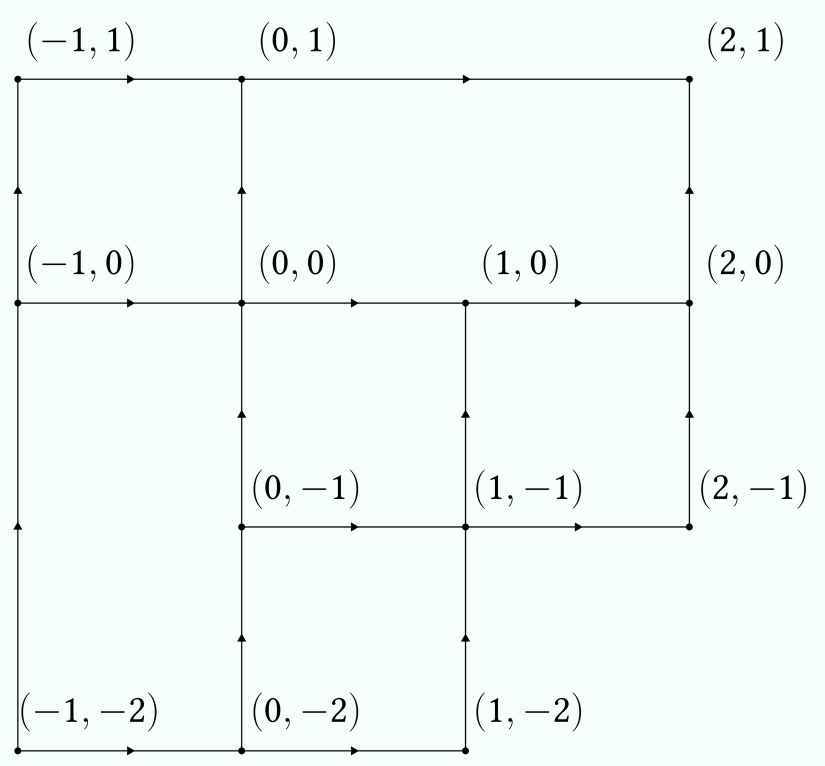
The map ψ is an isomorphism of posets from $\mathcal{TI}_n$ to $\mathcal{CC}_n$.

$$\begin{array}{ccc} \mathcal{TID}_n & \xrightarrow{\chi} & \mathcal{IP}_n \\ \uparrow \phi & & \downarrow \rho \\ \mathcal{CC}_n & \xleftarrow{\psi} & \mathcal{TI}_n \end{array}$$

Cubic realization

All cubic coordinates of size n can be placed in the space $\mathbb{R}^{n-1}$, as space coordinates. We connect a cubic coordinates c to an other c' with an arrow if and only if $c \lessdot c'$.

The oriented graph thus obtained is the cubic realization of cubic coordinate lattice.



Example

Cubic realization in $\mathbb{R}^2$ of $\mathcal{CC}_3$.

On these realizations, we can see a formation of cells. We want to give a combinatorial characterization of this cells in order to understand their arrangement.

To do that, we need to describe for each cell the cubic coordinate covered by $n - 1$ elements and the cubic coordinate covering $n - 1$ elements.

Let $c \in \mathcal{CC}_n$. Suppose that there is $c' \in \mathcal{CC}_n$ such that $c'_i > c_i$ and $c'_j = c_j$ for all $j \neq i$, with $i, j \in [n-1]$. We define then the map of minimal increase $\uparrow_i$ by

$$\uparrow_i(c) = (c_1, \dots, c_{i-1}, \widehat{c_i}, c_{i+1}, \dots, c_{n-1}),$$

such that $c \lessdot \uparrow_i(c)$ and $c_i < \widehat{c_i} \leqslant c'_i$.

Let $c^m \in \mathcal{CC}_n$, then c^m is minimal-cellular if for all $i \in [n-1]$, $\uparrow_i(c^m)$ is a cubic coordinate.

If c^m is minimal-cellular, then by making minimal increase on its entries from the right to the left, we always obtain a cubic coordinate.

Let $c^M \in \mathcal{CC}_n$, then c^M is the maximal-cellular correspondent of c^m if

$$c^M = \uparrow_1(\uparrow_2(\dots(\uparrow_{n-1}(c^m))\dots)).$$

We denote by $\langle c^m, c^M \rangle$ the corresponding cell.

Example

$c^m = (0, -1, 1, -1, -5, 0, 1, -1, -3)$ is minimal-cellular, and its maximal-cellular correspondent is $c^M = (1, 0, 2, 0, -4, 3, 2, 0, -2)$.

Let γ be the map defined for all $i \in [n-1]$ by

$$\gamma(c_i^m, c_i^M) = \begin{cases} c_i^m & \text{if } c_i^m < 0, \\ c_i^M & \text{if } c_i^m \geqslant 0, \end{cases}$$

and Γ be the map from the set of cells of size n to the set of $(n-1)$-tuples defined by

$$\Gamma(\langle c^m, c^M \rangle) = (\gamma(c_1^m, c_1^M), \gamma(c_2^m, c_2^M), \dots, \gamma(c_{n-1}^m, c_{n-1}^M)).$$

Example

The cell $\langle (0, -1, 1, -1, -5, 0, 1, -1, -3), (1, 0, 2, 0, -4, 3, 2, 0, -2) \rangle$ is sent to $(1, -1, 2, -1, -5, 3, 2, -1, -3)$.

Theorem [C., 2019]

The map Γ is a bijection from the set of cells of size n to $\mathcal{CC}_n^{sync}$.

This result allows us to compute a volume of the cubic realization, and to count the number of cells.

Another result obtained thanks to the isomorphism of posets ψ, is a generalization of results in Tamari lattices [Björner, Wachs, 1997]. Indeed, we can give a edge-labeling λ similar to the EL-labeling of Tamari posets.

Theorem [C., 2019]

The map λ gives an EL-labeling of $\mathcal{CC}_n$. Moreover, there is at most one falling chain in each interval of $\mathcal{CC}_n$.

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