Refining the bijections among ascent sequences, (2+2)-free posets integer matrices and pattern-avoiding permutations

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Background

The combined work of Bousquet-Mélou, Claesson, Dukes, Jelínek, Kitaev, Kubitzke and Parviainen has resulted in non-trivial bijections among ascent sequences, (2+2)-free posets, upper-triangular integer matrices, and pattern-avoiding permutations. To probe the finer behavior of these bijections, we study two types of restrictions on ascent sequences. These restrictions are motivated by our results that their images under the bijections are natural and combinatorially significant. In addition, for one restriction, we are able to determine the effect of poset duality on the corresponding ascent sequences, matrices and permutations, thereby answering a question of the first author and Parviainen in this case. The second restriction should appeal to Catalaniacs.

Four combinatorial objects and restrictions

Asc: Ascent sequences are sequences of non-negative integers $a = (a_1, \ldots, a_n)$ such that $a_{i+1} \in [0, 1 + \mathsf{asc}(a_1, \ldots, a_i)]$.

RAsc: Ascent sequences such that $a_{i+1} \in [0, a_i] \cup \{1 + \mathsf{asc}(a_1, \ldots, a_i)\}$.

CAsc: Ascent sequences avoiding the sequence $(1, 0, 1)$.

Posets: (2+2)-free posets, also known as *interval orders*, are characterised by having no induced subposet isomorphic to a disjoint union of two 2-element chains.

RPosets: Those posets $P \in$ Posets having a chain of length $\ell(P)$.

CPosets: Series parallel posets P.

Matrices: Upper-triangular matrices having non-negative entries and containing no rows or columns consisting of only zeros.

RMatrices: Diagonal entries are all positive.

CMatrices: A "SE-free" matrix.

Perms: Permutations avoiding $2|3\bar{1}$

RPerms: Permutations avoiding $3\bar{1}52\bar{4}$.

CPerms: Permutations avoiding 231.

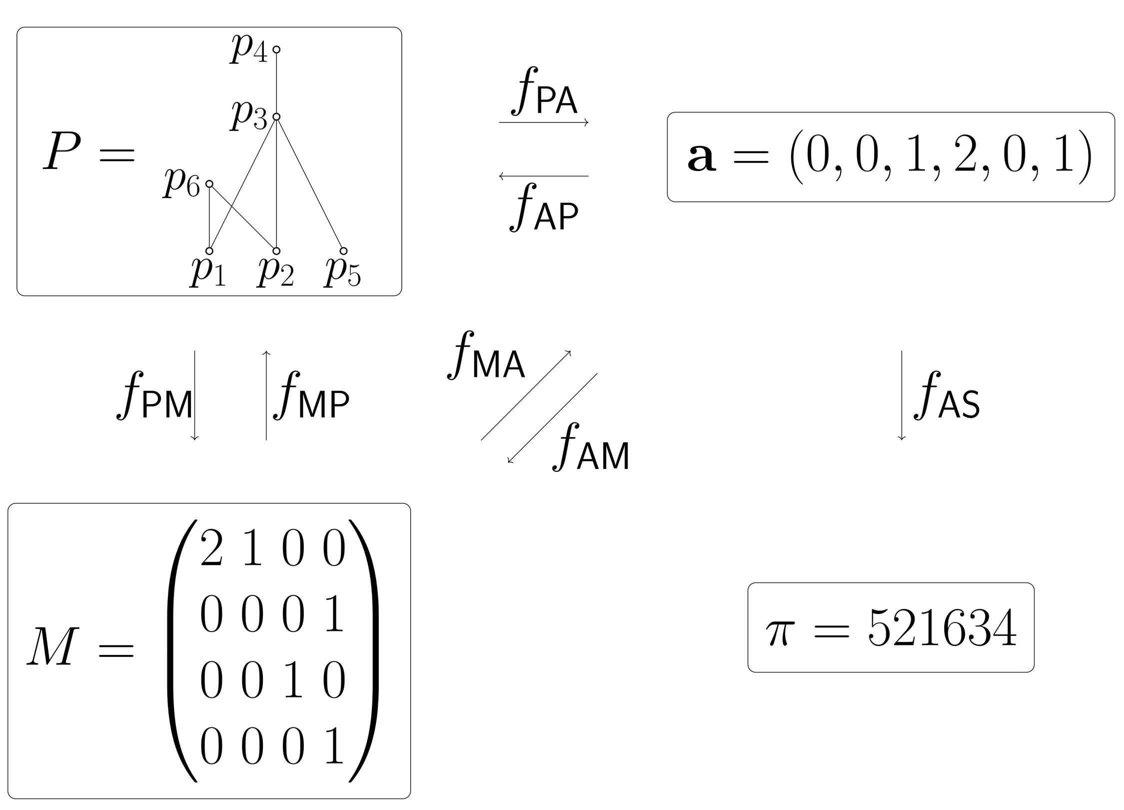
Bijections

Bijections between the four main sets defined above have been given in the papers [1, 4]. We will use the notation f_{AM} to denote the bijection from Asc to Matrices, and in general the subscripts of f indicate the first letter of the domain set and the first letter of the range. The bijections from the set Asc to the other sets have been explicitly stated in the literature. Each of these bijections are, in essence, recursive constructions, and have three different rules depending on the value of a_{i+1}.

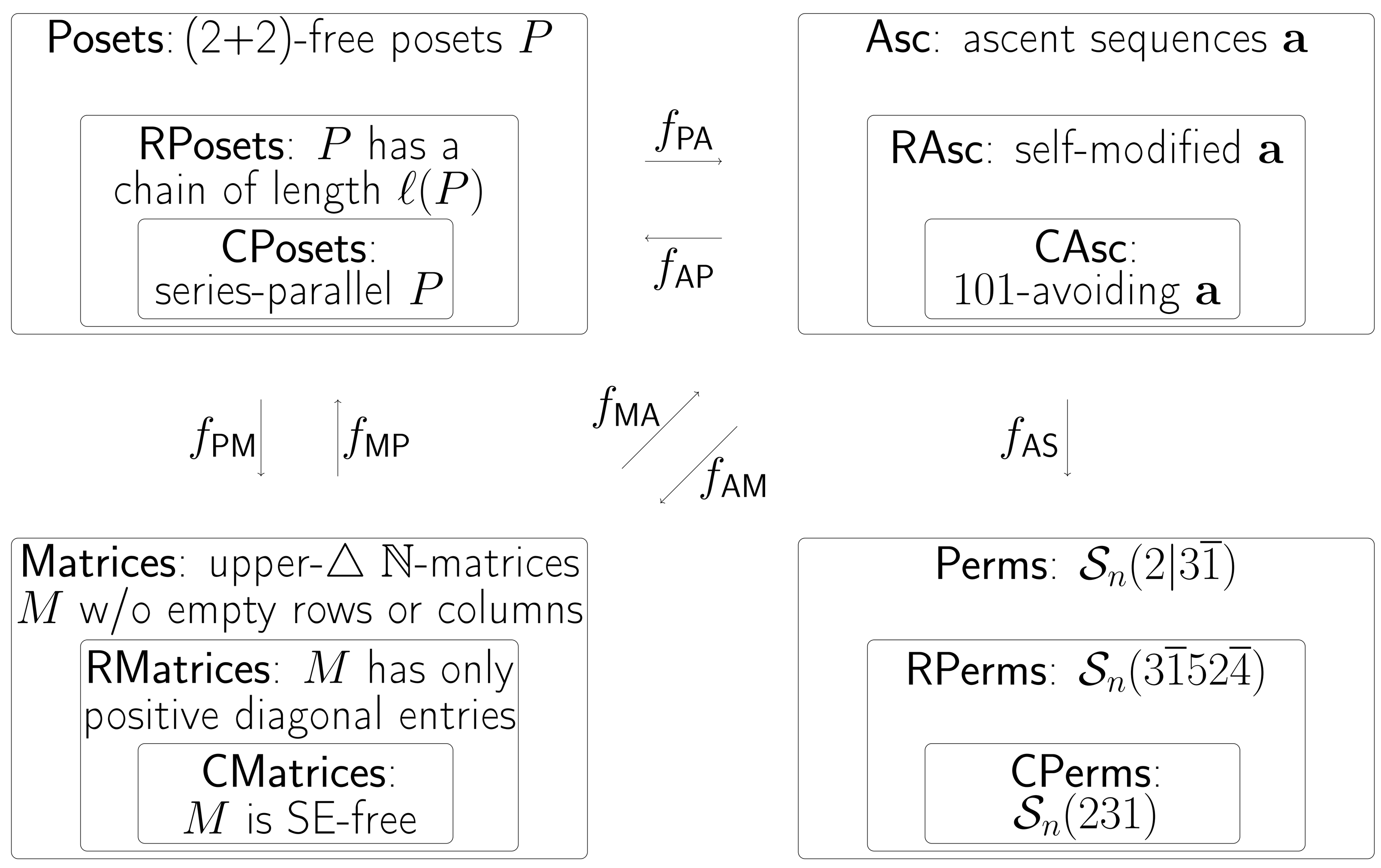
If $a = (a_1, \ldots, a_n) \in$ Asc, then

$$f_{AX}(a_1, \ldots, a_{i+1})) \leftarrow \begin{cases} \mathsf{action}_1(a_{i+1}, f_{AX}(a_1, \ldots, a_i)) & \text{if } a_{i+1} \in [0, a_i] \\ \mathsf{action}_2(a_{i+1}, f_{AX}(a_1, \ldots, a_i)) & \text{if } a_{i+1} = 1 + \mathsf{asc}(a_1, \ldots, a_i) \\ \mathsf{action}_3(a_{i+1}, f_{AX}(a_1, \ldots, a_i)) & \text{if } a_{i+1} \in (a_i, \mathsf{asc}(a_1, \ldots, a_i)) \end{cases}$$

The recursive constructions involving only action_1 and action_2 give rise to more regular structures.



Theorem: The bijections, when restricted to the sets outlined above, are bijections between the following families:



Duality

Theorem: Suppose $\mathbf{a} \in \mathsf{RAsc}_n$ with

$$\mathbf{a} = (0^{m_{11}}, 1^{m_{22}}, 0^{m_{12}}, 2^{m_{33}}, 1^{m_{23}}, 0^{m_{13}}, \ldots, (d-1)^{m_{dd}}, (d-2)^{m_{d-1d}}, \ldots, 0^{m_{1d}})$$

The dual ascent sequence $\mathbf{a}^*$ according to f_{PA} is given by

$$\mathbf{a}^* = (0^{m_{dd}}, 1^{m_{d-1d-1}}, 0^{m_{d-1d}}, 2^{m_{d-2d-2}}, 1^{m_{d-2d-1}}, 0^{m_{d-2d}}, \ldots, (d-1)^{m_{11}}, (d-2)^{m_{12}}, \ldots, 0^{m_{1d}}).$$

Theorem: Let $\pi \in \mathcal{S}_n(3\bar{1}52\bar{4}) =$ RPerms. The dual permutation according to f_{PS} is given by $\pi^* = (\mathrm{comp}(\mathrm{rev}(\pi)))^{-1}$.

Catalan restriction and series-parallel posets

The four sets CPerms, CAsc, CMatrices, CPosets are in one-to-one correspondence with one-another, and are enumerated by the Catalan numbers.

- The set CPosets is the set of *series parallel posets*, those posets that are both (2+2)-free and N-free.
- An SE-pair of a matrix $M \in$ Matrices is a pair of non-zero entries m_{ij} and $m_{i'j'}$ such that $i < i'$, $j < j'$ and $i' < j$. We say that M is *SE-free* if it contains no SE-pair and let CMatrices be the set of SE-free matrices in Matrices.

References

[1] M. Bousquet-Mélou, A. Claesson, M. Dukes and S. Kitaev. $(\mathbf{2}+\mathbf{2})$-free posets, ascent sequences and pattern avoiding permutations. *J. Comb. Theory Ser. A* (2010).

[2] A. Claesson, M. Dukes and M. Kubitzke. Partition and composition matrices. *J. Comb. Theory Ser. A* (2011).

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[4] M. Dukes and R. Parviainen. Ascent sequences and upper triangular matrices containing non-negative integers. *Electron. J. Comb.* (2010).

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