TRIANGULATIONS OF THE PRODUCT OF SPHERES WITH FEW VERTICES

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INTRODUCTION

Let C_d^* denote the d-**cross-polytope** with d pairs of antipodal vertices $\{x_i, y_i\}$ for $1 \le i \le d$. Define a **switch** to be an occurrence of $x_i y_{i+1}$ or $y_i x_{i+1}$ in the ordered vertices of a face. Let $B(i,d)$ be the subcomplex of ∂C_d^* constructed by taking all facets of ∂C_d^* with at most i switches. Let Γ_i be the union of facets in ∂C_d^* with at most 2 switches and with j y-labels.

Let Δ be a $(d-1)$-dimensional simplicial complex.

- Δ is said to be a **combinatorial manifold** provided that the link of every non-empty face σ of Δ is a triangulated $(d-|\sigma|-1)$-dimensional PL ball or sphere. A combinatorial ball (resp., sphere) is a combinatorial manifold which triangulates a ball (resp., sphere).
- Δ is said to be **centrally symmetric**, or **cs**, if it is endowed with a free involution $\alpha : V(\Delta) \to V(\Delta)$ that induces a free involution on the set of all non-empty faces.
- Δ is said to be **balanced** if the graph of Δ is d-colorable; that is, there exists a coloring map $\kappa : V \to \{1, 2, \cdots, d\}$ such that $\kappa(x) \neq \kappa(y)$ for all edges $\{x, y\} \in \Delta$.

PREVIOUS RESULTS

It is known that for $i \le j$, the minimal triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ requires at least $i + 2j + 4$ vertices, see [1]. However, this lower bound is not always *tight* - currently, the smallest triangulations are constructed in [2].

Theorem 1 *There exists a centrally symmetric 2d-vertex triangulation of* $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$ *obtained by taking the boundary complex of* $B(i,d)$.

Some lower-dimensional minimal triangulations, such as $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for $d \le 6$ and $\mathbb{S}^3 \times \mathbb{S}^3$, are found by the computer program BISTELLAR [3]. In the specific case of balanced triangulations, [4] gives us the following.

Theorem 2 *For all* $d \ge 3$, *there exists a balanced simplicial manifold that triangulates* $\mathbb{S}^{d-2} \times \mathbb{S}^1$ *with* $3d$ *vertices for odd* d, *and* $3d+2$ *vertices for even* d, *and a balanced simplicial manifold that triangulates the nonorientable* $\mathbb{S}^{d-2}$*-bundle over* $\mathbb{S}^1$ *with* $3d$ *vertices for even* d, *and* $3d+2$ *vertices for odd* d.

CS TRIANGULATIONS OF SPHERE PRODUCTS

Our inductive construction is based on the following proposition.

Proposition 3 *Fix* d *and* $i \le \frac{d-1}{2}$. *Let* D_1 *and* D_2 *be two combinatorial d-balls such that*

1. $\partial(D_1 \cup D_2)$ *is a combinatorial* $(d-1)$*-manifold in a combinatorial d-sphere.*
2. $D_1 \cap D_2 = \partial D_1 \cap \partial D_2$ *is a path-connected combinatorial* $(d-1)$*-manifold (with boundary) that has the same homology as* $\mathbb{S}^{i-1}$.
3. $\partial(D_1 \cap D_2)$ *has the same homology as* $\mathbb{S}^{i-1} \times \mathbb{S}^{d-i-1}$.

Then $\partial(D_1 \cup D_2)$ *triangulates* $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ *for* $d \ge 5$.

Main Result 1. Let $d = 2m+1$. Define

$$D_1 = (\cup_{k=0}^{m+1} \Gamma_k) * \{x_{d+1}\} \qquad D_2 = (\cup_{k=m}^{d} \Gamma_k) * \{y_{d+1}\}$$

with the case when d is even defined similarly, and $*$ denoting the join operation. By Proposition 3 and Theorem 1, this choice of D_1 and D_2 give another centrally symmetric triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$.

Remark 4 *Using only two additional vertices, we can build a cs triangulation of* $\mathbb{S}^{i+1} \times \mathbb{S}^{d-i-1}$ *from a cs triangulation of* $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$, *and thus can inductively construct a family of triangulations of* $\mathbb{S}^j \times \mathbb{S}^{d-i-1}$ *where* $j \le d-i-1$.

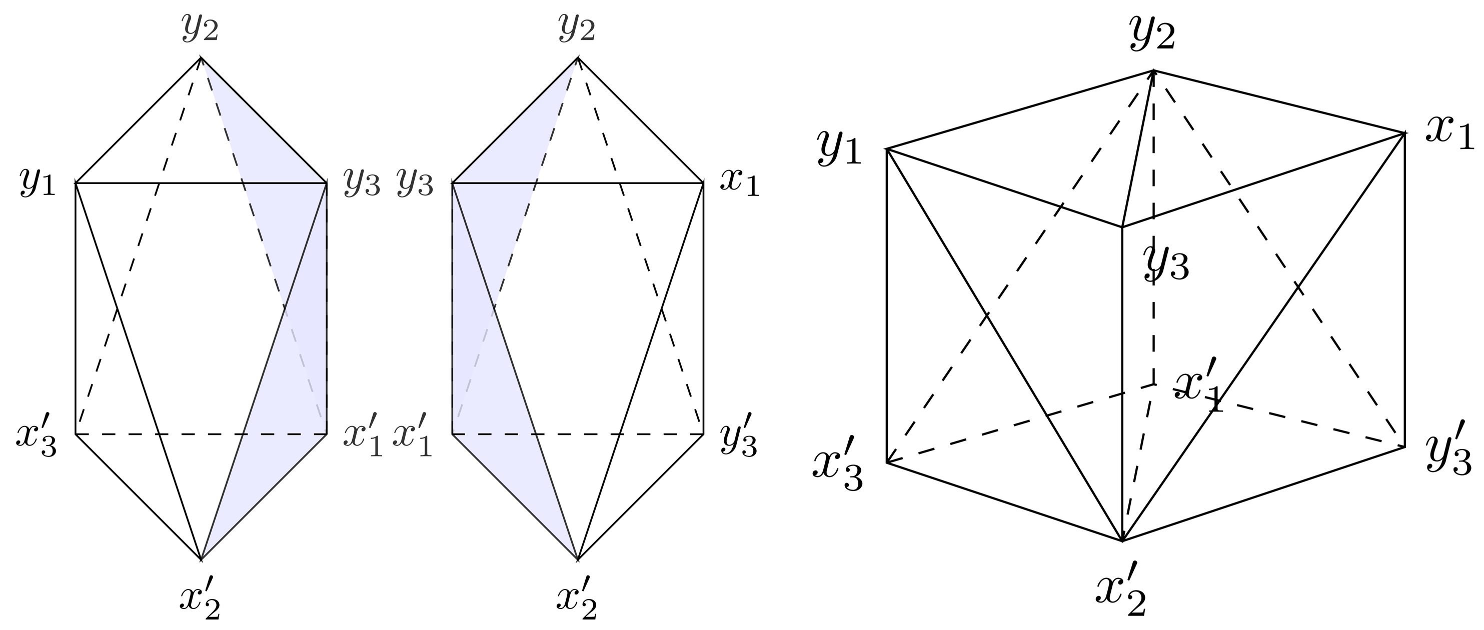
BALANCED TRIANGULATION OF $\mathbb{S}^2 \times \mathbb{S}^{d-3}$

Strategy. We note that $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ admits the decomposition

$$(\mathbb{D}^2 \times \mathbb{S}^{d-3}) \cup (\mathbb{D}^2 \times \mathbb{S}^{d-3})$$

Let B, B' be two copies of $B(d-3,d)$ in distinct cross-polytopes; they have the same homology as $\mathbb{S}^{d-3}$. Our method is to find a balanced combinatorial manifold N such that $\|N\| \cong \|\partial B\| \times \mathbb{D}^1$ with $\partial N = \partial B \sqcup \partial B'$. We then prove that $B \cup N \cup B'$ triangulates $\mathbb{S}^2 \times \mathbb{S}^{d-3}$.

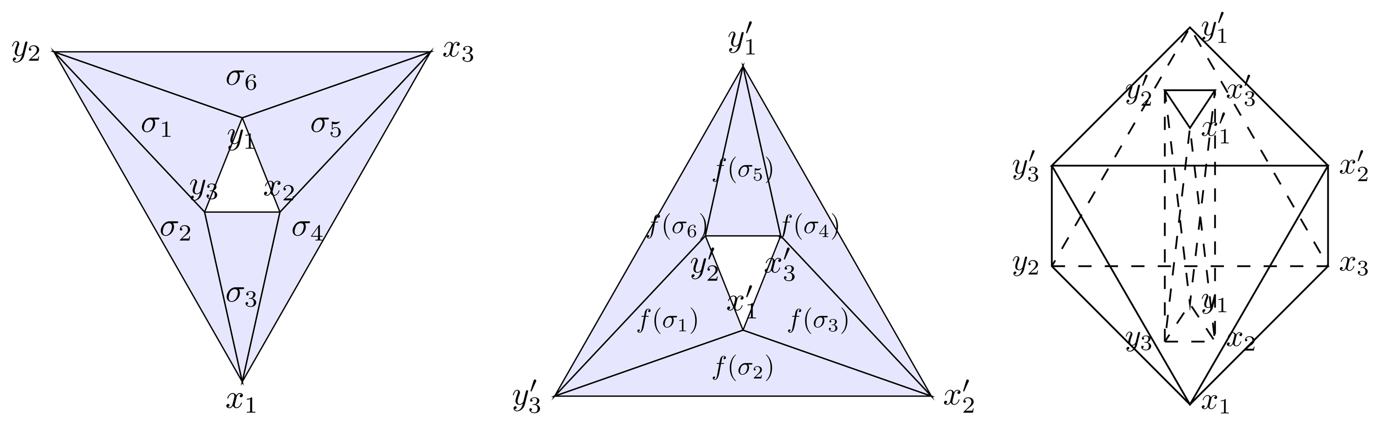
To find a suitable choice of N, we define a variant of the usual connected sum, called the $\Diamond$**-connected sum**, which applies to cross-polytopes rather than simplices, and preserves the balancedness of the resulting construction.



Main Result 2. We construct a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ as follows.

1. Take $B = B(1,d)$.
2. Choose an appropriate simplicial automorphism f on B.
3. Extend every pair of facets $\sigma \in B$ and $f(\sigma) \in f(B)$ to a d-cross-polytope, and build the $\Diamond$-connected sum of these cross-polytopes; take N as its "side".
4. Let $\Sigma = B^c \cup N \cup f(B)^c$, where B^c is the complement of B in ∂C_d^*.

Then, Σ is a $4d$-vertex balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$. Furthermore, it admits a vertex-transitive action of $\mathbb{Z}_2 \times \mathcal{D}_{2d}$.



The above shows B, $f(B)$, and the construction Σ for the case of $d = 3$.

Working with Lorenzo Venturello, we developed a program to attempt to reduce the number of vertices through exhaustive search.

Problem 5 *Find a small balanced triangulation of* $\mathbb{S}^i \times \mathbb{S}^j$ *for all* $1 \le i \le j$.

Problem 6 *Determine the sharp lower bound on the number of vertices required for a balanced triangulation of* $\mathbb{S}^i \times \mathbb{S}^j$ *for all* $1 \le i \le j$.

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