The equivariant volumes of the permutahedron

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Abstract

We consider the action of the symmetric group S_n on the permutahedron Π_n. We prove that if σ is a permutation of S_n which has m cycles of lengths $l_1, \dots, l_m$, then the subset of Π_n fixed by σ is a polytope with normalized volume $n^{m-2}\gcd(l_1, \dots, l_m)$.

Introduction

The n-permutahedron is the polytope in $\mathbb{R}^n$ whose vertices are the permutations of $[n]$:

$$\Pi_n := \operatorname{conv}\{(\pi(1), \pi(2), \dots, \pi(n)) : \pi \in S_n\}.$$

The symmetric group S_n acts on $\Pi_n \subset \mathbb{R}^n$ by permuting coordinates; more precisely, a permutation $\sigma \in S_n$ acts on a point $x = (x_1, x_2, \dots, x_n) \in \Pi_n$, by

$$\sigma \cdot x := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(n)}).$$

Our main result is a generalization of the fact, due to Stanley, that $\operatorname{Vol}\Pi_n = n^{n-2}$

Definition 1: The *fixed polytope of the permutahedron* Π_n *under a permutation* σ *of* $[n]$ is

$$\Pi_n^\sigma = \{x \in \Pi_n : \sigma \cdot x = x\}.$$

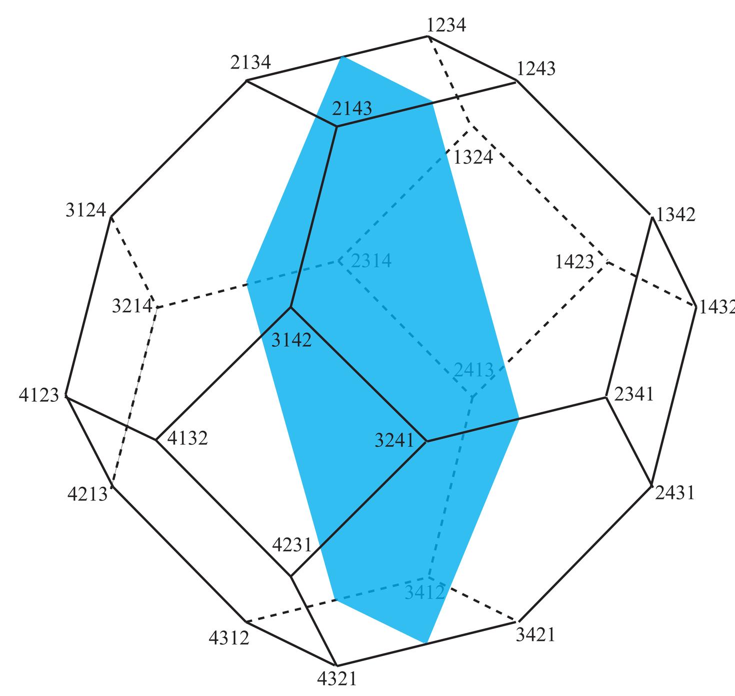

Figure 1. The fixed polytope $\Pi_4^{(12)}$ of the permutahedron Π_4 under $(12) \in S_4$ is a hexagon.

Normalizing the volume

The permutahedron and its fixed polytopes are not full-dimensional. We normalize volumes so that every primitive parallelotope has volume 1. This is the normalization under which the volume of Π_n equals n^{n-2}.

More precisely, let P be a d-dimensional polytope on an affine d-plane $L \subset \mathbb{Z}^n$. Assume L is integral, in the sense that $L \cap \mathbb{Z}^n$ is a lattice translate of a d-dimensional lattice Λ. We call a lattice d-parallelotope in L *primitive* if its edges generate the lattice Λ; all primitive parallelotopes have the same volume. Then we define the volume of a d-polytope P in L to be $\operatorname{Vol}(P) := \operatorname{EVol}(P)/\operatorname{EVol}(\square)$ for any primitive parallelotope $\square$ in L, where EVol denotes Euclidean volume.

Describing the fixed polytopes of the permutahedron

Lemma 1: The volume of Π_n^σ only depends on the cycle type of σ.

Definition 2: Given $\sigma \in S_n$, we say a permutation $v = (v_1, \dots, v_n)$ of $[n]$ is σ-*standard* if it satisfies the following property: for each cycle $\sigma_j = (j_1\ j_2\ \cdots\ j_r)$ of σ, $(v_{j_1}, v_{j_2}, \dots, v_{j_r})$ is a sequence of consecutive integers in increasing order.

We define the set of σ-*vertices* to be:

$$\operatorname{Vert}(\sigma) = \{\overline{w} : w \text{ is a } \sigma\text{-standard permutation of } [n]\}.$$

Definition 3: For any $w \in \mathbb{R}^n$, the average of the σ-orbit of w is

$$\overline{w} = \sum_{k=1}^{m} \frac{\sum_{j \in \sigma_k} w_j}{l_k} e_{\sigma_k}.$$

Corollary 1: The set $\operatorname{Vert}(\sigma)$ of σ-vertices consists of the $m!$ points

$$\overline{v_\prec} := \sum_{k=1}^{m} \left(\frac{l_k + 1}{2} + \sum_{j\,:\,\sigma_j \prec \sigma_k} l_j \right) e_{\sigma_k}$$

as $\prec$ ranges over the $m!$ possible linear orderings of $\sigma_1, \sigma_2, \dots, \sigma_m$.

Definition 4: Let M_σ denote the Minkowski sum

$$M_\sigma = \sum_{1 \le j < k \le m} [l_j e_{\sigma_k}, l_k e_{\sigma_j}] + \sum_{k=1}^{m} \frac{l_k + 1}{2} e_{\sigma_k}.$$

Theorem 1 [Ardila, Schindler, ARVM]

Let σ be a permutation of $[n]$ whose cycles $\sigma_1, \dots, \sigma_m$ have respective lengths $l_1, \dots, l_m$. The fixed polytope Π_n^σ can be described as in Definition 1 and:

1. (Inequalities)
   - $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$,
   - for any proper subset $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$,
     $$x_{i_1} + x_{i_2} + \cdots + x_{i_k} \le 1 + 2 + \cdots + k, \text{ and}$$
   - for any i and j which are in the same cycle of σ, $x_i = x_j$.
2. (Vertex) the convex hull of the set $\operatorname{Vert}(\sigma)$, and
3. the Minkowski Sum from Definition 4.

Consequently, Π_n^σ is a zonotope that is combinatorially isomorphic to the permutahedron Π_m. It is $(m-1)$-dimensional and every σ-vertex is indeed a vertex of Π_n^σ.

The volumes of the fixed polytopes

We use the zonotope description of Π_n^σ to compute its volume, recalling that a zonotope can be tiled by parallelotopes as follows. If A is a set of vectors, then $B \subseteq A$ is called a *basis* for A if B is linearly independent and $\operatorname{rank}(B) = \operatorname{rank}(A)$. We define the parallelotope $\square B$ to be the Minkowski sum of the segments in B, that is,

$$\square B := \left\{ \sum_{b \in B} \lambda_b b \,:\, 0 \le \lambda_b \le 1 \text{ for each } b \in B \right\}.$$

Theorem: Let $A \subset \mathbb{Z}^n$ be a set of lattice vectors of rank d.

1. The zonotope $Z(A)$ can be tiled using one translate of the parallelotope $\square B$ for each basis B of A. Therefore, the volume of the d-dimensional zonotope $Z(A)$ is
   $$\operatorname{Vol}(Z(A)) = \sum_{\substack{B \subseteq A \\ B \text{ basis}}} \operatorname{Vol}(\square B).$$
2. For each $B \subset \mathbb{Z}^n$ of rank d, $\operatorname{Vol}(\square B)$ equals the index of $\mathbb{Z}B$ as a sublattice of $(\operatorname{span} B) \cap \mathbb{Z}^n$. Using the vectors in B as the columns of an $n \times d$ matrix, $\operatorname{Vol}(B)$ is the greatest common divisor of the minors of rank d.

By Theorem 1, Π_n^σ is a translate of the zonotope generated by the set

$$F_\sigma = \{l_k e_{\sigma_j} - l_j e_{\sigma_k} ; 1 \le j < k \le m\}.$$

This set of vectors has a nice combinatorial structure, allowing us to describe the bases B and the volumes $\operatorname{Vol}(\square B)$ combinatorially. For a tree T whose vertex set is $[m]$, let

$$F_T = \{l_k e_{\sigma_j} - l_j e_{\sigma_k} \,:\, j < k \text{ and } jk \text{ is an edge of } T\},$$

$$E_T = \left\{ \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} \,:\, j < k \text{ and } jk \text{ is an edge of } T \right\}.$$

Lemma 2: The vector configuration

$$F_\sigma := \{l_k e_{\sigma_j} - l_j e_{\sigma_k} \,:\, 1 \le j < k \le m\}$$

has m^{m-2} bases: they are the sets F_T as T ranges over the spanning trees on $[m]$.

Lemma 3: For any tree T on $[m]$ we have

$$1. \quad \operatorname{Vol}(\square F_T) = \prod_{i=1}^{m} l_i^{\deg_T(i)} \operatorname{Vol}(E_T),$$

$$2. \quad \operatorname{Vol}(\square E_T) = \frac{\gcd(l_1, \dots, l_m)}{l_1 \cdots l_m},$$

where $\deg_T(i)$ is the number of edges containing vertex i in T.

Lemma 4: For any positive integer $m \ge 2$ and unknowns $x_1, \dots, x_m$, we have

$$\sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} x_i^{\deg_T(i)-1} = (x_1 + \cdots + x_m)^{m-2}.$$

Theorem 2 [Ardila, Schindler, ARVM]

If σ is a permutation of $[n]$ whose cycles have lengths $l_1, \dots, l_m$, then the normalized volume of the fixed polytope of Π_n under σ is

$$\operatorname{Vol}\Pi_n^\sigma = n^{m-2}\gcd(l_1, \dots, l_m).$$

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