Refined face count in uniform triangulations of the Legendre polytope

Richard Ehrenborg *University of Kentucky*, Gábor Hetyei† *Univeristy of North Carolina at Charlotte*, Margaret Readdy *University of Kentucky*

The Legendre polytope

The *Legendre polytope* $P_n \subset \mathbb{R}^{n+1}$ is the convex hull of all pairwise differences $e_j - e_i$ for an orthonormal basis $\{e_1, e_2, \ldots, e_{n+1}\}$. We may represent each vertex with a forward or backward arrow $i \longrightarrow j$. A set of arrows is affine independent if no head is also a tail, and the underlying undirected graph contains no cycle. [2]. For unimodality reasons, all triangulations of the boundary with no added vertices have the same face numbers f_j. The polynomial $P_n(x) = \sum_j f_j((x-1)/2)^j$ is the *nth Legendre polynomial* [2]. Every 2-face of P_n is either a triangle, formed by 3 arrows with a common head or a common tail, or it is a square, formed by 4 arrows on the same pairs of 2 heads and 2 tails [1]. All pulling triangulations of the boundary are *flag* [1].

Uniform flag triangulations

A flag simplicial complex $\triangle_n$ on the set of arrows is a *uniform flag complex* if determining whether or not a pair of vertices $\{(i_1, j_1), (i_2, j_2)\}$ forms an edge depends only on the equalities and inequalities between the values of i_1, i_2, j_1 and j_2. We refer to the relative order of the heads and tails by a word consisting of two letters H and two letters T. For example the statement "$THTH$ type pairs of arrows do not nest" means that, for $i_1 < j_1 < i_2 < j_2$, the pair $\{(i_1, j_1), (i_2, j_2)\}$ is an edge, but the pair $\{(i_1, j_2), (i_2, j_1)\}$ is not. These triangulations satisfy the following necessary conditions:

1. No node is a head and a tail simultaneously
2. The complex contains all triangular 2-faces of P_n
3. For every square 2-face of P_n, exactly one of its diagonals cuts the square into two triangles belonging to the triangulation.

To define a uniform flag triangulation, we only need to describe which pairs of arrows, with disjoint sets of heads and tails, are edges. There are at most 2^6 uniform flag triangulations for a fixed n, as there are 6 ways to line up two heads and 2 tails of a pair of arrows, forming an edge.

We describe all flag triangulations and compute their *refined face polynomial* in which the coefficient of $x^i y^j$ is the number of faces with i forward arrows and j backward arrows. $F(\mathcal{F}, x, y, t) = \sum_{n,i,j \geq 0} f(\mathcal{F}_n, i, j) \cdot x^i y^j t^n$ counts the number of faces in P_n, having i forward arrows and j backward arrows. $F(\widehat{\mathcal{F}}, x, y, t)$ does the same for the *saturated faces*, defined as sets of arrows where every node in $\{1, 2, \ldots, n+1\}$ is an endpoint of an arrow. These generating functions are connected by the following formulas:

$$F(\mathcal{F}, x, y, t) = -\frac{t}{(1-t)^2} \cdot \delta_{\mathcal{F}_0, \{\emptyset\}} + \frac{1}{(1-t)^2} \cdot F\left(\widehat{\mathcal{F}}, x, y, \frac{t}{1-t}\right),$$

$$F(\widehat{\mathcal{F}}, x, y, z) = \frac{z}{1+z} \cdot \delta_{\widehat{\mathcal{F}}_0, \{\emptyset\}} + \frac{1}{(1+z)^2} \cdot F\left(\mathcal{F}, x, y, \frac{z}{1+z}\right).$$

Our main result

Let $\triangle_n$ be a uniform flag complex on the vertex set V_n for some $n \geq 5$ that satisfies the above stated necessary conditions. Then the complex $\triangle_n$ represents a triangulation of the boundary ∂P_n of the Legendre polytope if and only if it satisfies exactly one of the following conditions:

1. Both $THTH$ and $HTHT$ types of pairs of arrows do not nest, and both $HTTH$ and $THHT$ types of arrows do not cross. This class contains the *lexicographic pulling triangulation*, and we call it the *lex class*.
2. Both $THTH$ and $HTHT$ types of pairs of arrows nest, and both $HTTH$ and $THHT$ types of arrows cross. This class contains the *revlex pulling triangulation* and we call it the *revlex class*.
3. Exactly one of the $THTH$ and $HTHT$ types of pairs of arrows nest. Furthermore, if both $THHT$ and $HTTH$ types of pairs cross then both $TTHH$ and $HHTT$ types of pairs nest. This class contains *the Simion type B associahedron*, and we call it the *Simion class*.

Proof: The proof of the necessity part is straightforward. To show sufficiency, we used a rephrased variant of a result of Oh and Yoo [3], describing all triangulations of a product of two simplices in terms of *matching ensembles*.

Refined face count in the Simion class

If $THTH$ types of pairs of arrows do not nest and $HTHT$ types of pairs of arrows nest then the $F(\widehat{\mathcal{F}}, x, y, z)$ equals

$$\frac{C(yz(z+1)) + z}{1+z} + \frac{xz \cdot (1 + zC(yz(z+1))) \cdot C(yz(z+1))^2}{(1+z) \cdot (1 - 2C(yz(z+1))xz - C(yz(z+1))^2 xz^2)}.$$

where $C(u) = \sum_{n \geq 0} C_n \cdot u^n = (1 - \sqrt{1-4u})/(2u)$ is the generating function of the Catalan numbers.

Facet count: The number $f(\triangle_n, 0, n)$ is C_n. For $i \geq 1$

$$f(\triangle_n, i, n-i) = 2^{i-1} \cdot \frac{(i+1) \cdot (2n-i)!}{(n-i)! \cdot (n+1)!} \quad \text{holds.}$$

The remaining triangulations in the class are obtained by reversing all arrows. For these, we must swap x and y, respectively i and $n-i$.

Refined face count in the revlex class

In this class we enumerate using a *node-enriched exponential generating function* for the saturated faces, in which each nonempty $\sigma \in \widehat{\mathcal{F}}_n$ contributes a term $x^i y^j \cdot u^{a+1} \cdot v^{b+1} \cdot t^n/((a+1)! \cdot (b+1)!)$, where i, respectively j, is the number of forward, respectively backward arrows, and $a+1$, respectively $b+1$ is the number of nodes that are left ends, respectively right ends of arrows. This generating function equals

$$1 + \frac{1}{z} \cdot \widetilde{D}(uz, vz, x) + \frac{1}{z} \cdot \widetilde{D}(vz, uz, y) + \frac{1}{z} \cdot \widetilde{D}(uz, vz, x) \cdot \widetilde{D}(vz, uz, y) + \frac{1}{z^2} \cdot \frac{\partial}{\partial u}\widetilde{D}(uz, vz, x) \cdot \frac{\partial}{\partial v}\widetilde{D}(vz, uz, y) + \frac{1}{z^2} \cdot \frac{\partial}{\partial v}\widetilde{D}(uz, vz, x) \cdot \frac{\partial}{\partial u}\widetilde{D}(vz, uz, y),$$

where $\widetilde{D}(u, v, x) = \sum_{a,b \geq 0} \frac{D_{a,b}(x) \cdot u^{a+1} \cdot v^{b+1}}{(a+1)! \cdot (b+1)!}$ is an exponential generating function of the *Delannoy polynomials* $D_{a,b}(x)$, defined as the total weight of all Delannoy paths from $(0,0)$ to (a,b), where each step contributes a factor of x.

Fun fact: The mixed second partial derivative of $\widetilde{D}(u, v, x)$ is given by

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}\widetilde{D}(u, v, x) = \exp(x \cdot (u+v)) \cdot I_0\left(2\sqrt{(x^2+x) \cdot uv}\right)$$

where $I_0(z)$ is the modified Bessel function of the first kind.

Facet count: For $1 \leq k \leq n-1$, the number $f(\triangle_n, k, n-k)$ is given by

$$\sum_{i=1}^{k} \sum_{j=1}^{n-k} \binom{k-1}{i-1} \cdot \binom{n-k-1}{j-1} \cdot \left[\binom{n-k+i-j}{i} \cdot \binom{k-i+j}{j} + \binom{n-k+i-j}{i-1} \cdot \binom{k-i+j}{j-1}\right].$$

We also have $f(\triangle_n, 0, n) = f(\triangle_n, n, 0) = 2^{n-1}$.

Refined face count in the lex class

In this class, the number of $(k-1)$-dimensional faces in $\triangle_n$ with a given number i of forward arrows is *independent of i*:

$$f(\triangle_n, i, k-i) = \frac{1}{k+1} \cdot \binom{n+k}{k} \cdot \binom{n}{k}.$$

The combinatorial reason behind this formula will be published separately.

Facet count: The number $f(\triangle_n, k, n-k)$ is C_n, regardless of k.

References

[1] R. Ehrenborg, G. Hetyei and M. Readdy, Simion's type B associahedron is a pulling triangulation of the Legendre polytope, *Discrete Comput. Geom.* **60** (2018), 98–114.

[2] G. Hetyei, Delannoy orthants of Legendre polytopes, *Discrete Comput. Geom.* **42** (2009), 705–721.

[3] S. Oh and H. Yoo, Triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ and tropical oriented matroids, arXiv:1311. 6772v1 [math.CO].

http://webpages.uncc.edu/ghetyei/ ghetyei@uncc.edu