# The CDE property for skew vexillary permutations

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## "Barely set-valued" tableaux product formula

A **"barely set-valued" Standard Young Tableau (SYT)** of shape $\lambda$ is like an SYT, but with entries $\{1, 2, \dots, |\lambda|+1\}$ and exactly one box with two numbers:

SYT of shape $2 \times 2$:

| Tableau | Row 1 | Row 2 |
|---|---|---|
| 1 | 1 2 | 3 4 |
| 2 | 1 3 | 2 4 |

Barely set-valued SYT of shape $2 \times 2$:

| Tableau | Row 1 | Row 2 |
|---|---|---|
| 1 | 1 2 | 3 4,5 |
| 2 | 1 3 | 2 4,5 |
| 3 | 1 2,3 | 4 5 |
| 4 | 1 4 | 2,3 5 |
| 5 | 1 2,4 | 3 5 |
| 6 | 1 3 | 2,4 5 |
| 7 | 1 2 | 3,4 5 |
| 8 | 1 3,4 | 2 5 |
| 9 | 1,2 3 | 4 5 |
| 10 | 1,2 4 | 3 5 |

### Theorem (Frame-Robinson-Thrall, 1954)

$$\#\ \mathrm{SYT\ of\ shape\ } \lambda = n! \cdot \prod_{u \in \lambda} \frac{1}{h(u)},$$

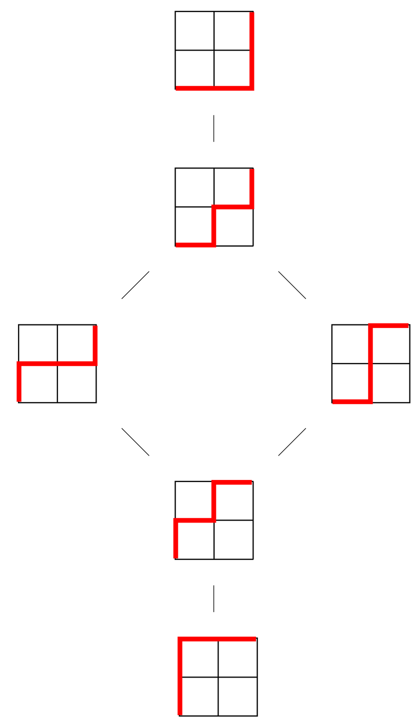
where $h(u)$ is the hook-length of the box $u \in \lambda$.

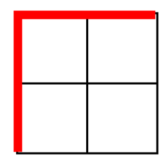
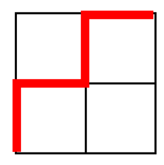
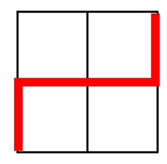
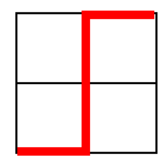
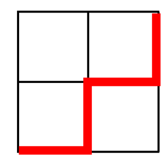
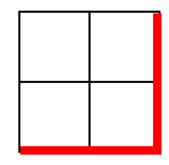
### Theorem (Chan-López Martín-Pfleuger-Teixidor i Bigas (CLPT), 2018)

$$\#\ \mathrm{barely\ set\text{-}valued\ SYT\ of\ sh.\ } a \times b = (ab+1) \cdot \frac{ab}{a+b} \cdot \#\mathrm{SYT\ of\ sh.\ } a \times b$$

## Coincidental Down-degree Expectations

Let $L :=$ poset of lattice paths in $a \times b$ rectangle ($\approx$ interval $[\varnothing, b^a]$ of Young's lattice):



| Lattice path $x$ |  |  |  |  |  |  |
|---|---|---|---|---|---|---|
| $\mathrm{uni}_L(x)$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |
| $\max_L(x)$ | $\frac{2}{10}$ | $\frac{2}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{2}{10}$ | $\frac{2}{10}$ |
| $\mathrm{ddeg}(x)$ | 0 | 1 | 1 | 1 | 2 | 1 |

Consider two probability distributions on $L$: the **uniform distribution** $\mathrm{uni}_L$; and the **maxchain distribution** $\max_L$, where $x \in L$ occurs proportional to the number of **maximal chains** ($\approx$ Standard Young Tableaux) containing $x$.

For $x \in L$, its **down-degree** is $\mathrm{ddeg}(x) := \#$ of elements of $L$ that $x$ covers.

**Reformulation of CLPT**: $\mathbb{E}(\max_L; \mathrm{ddeg}) = ab/(a+b)$.

(Context for CLPT: computing the genus of a moduli space in Brill-Noether theory.)

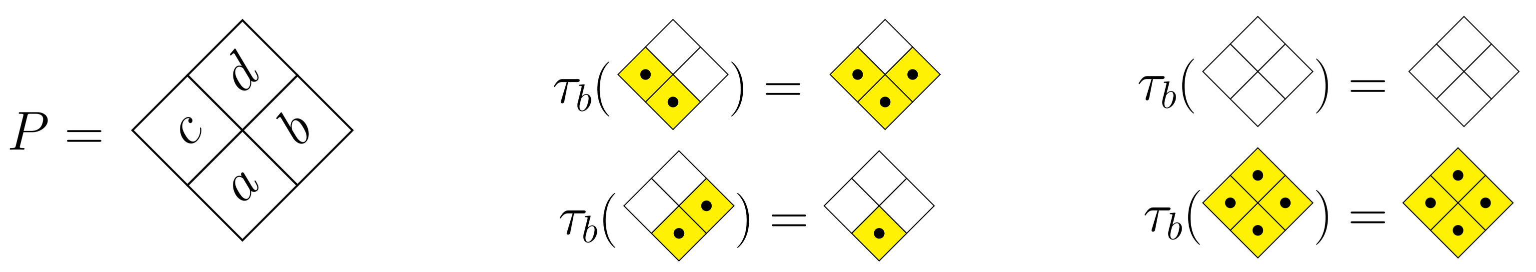
**A curious coincidence**: $\mathbb{E}(\mathrm{uni}_L; \mathrm{ddeg}) = ab/(a+b)$.

## Main Definition: the CDE property (Reiner-Tenner-Yong (RTY), 2018)

Poset $L$ has the coincidental down-degree expectations property (or "is CDE") if $\mathbb{E}(\max_L; \mathrm{ddeg}) = \mathbb{E}(\mathrm{uni}_L; \mathrm{ddeg})$. Note that $\mathbb{E}(\mathrm{uni}_L; \mathrm{ddeg})$ is the edge density of $L$.

## Toggling in a distributive lattice & the tCDE property

$J(P) :=$ **distributive lattice of order ideals** of a poset $P$. For $p \in P$, **toggling at** $p$ is the involution $\tau_p \colon J(P) \to J(P)$ that adds or removes $p$, if possible:




A probability distribution $\mu$ on $J(P)$ is **toggle-symmetric** if

$$\mathbb{P}(\mu; \text{can toggle in } p) = \mathbb{P}(\mu; \text{can toggle out } p) \text{ for all } p \in P.$$

We say $J(P)$ is **toggle CDE (tCDE)** if $\mathbb{E}(\mu; \mathrm{ddeg}) = \mathbb{E}(\mathrm{uni}_{J(P)}; \mathrm{ddeg})$ for any toggle-symmetric distribution $\mu$ on $J(P)$.

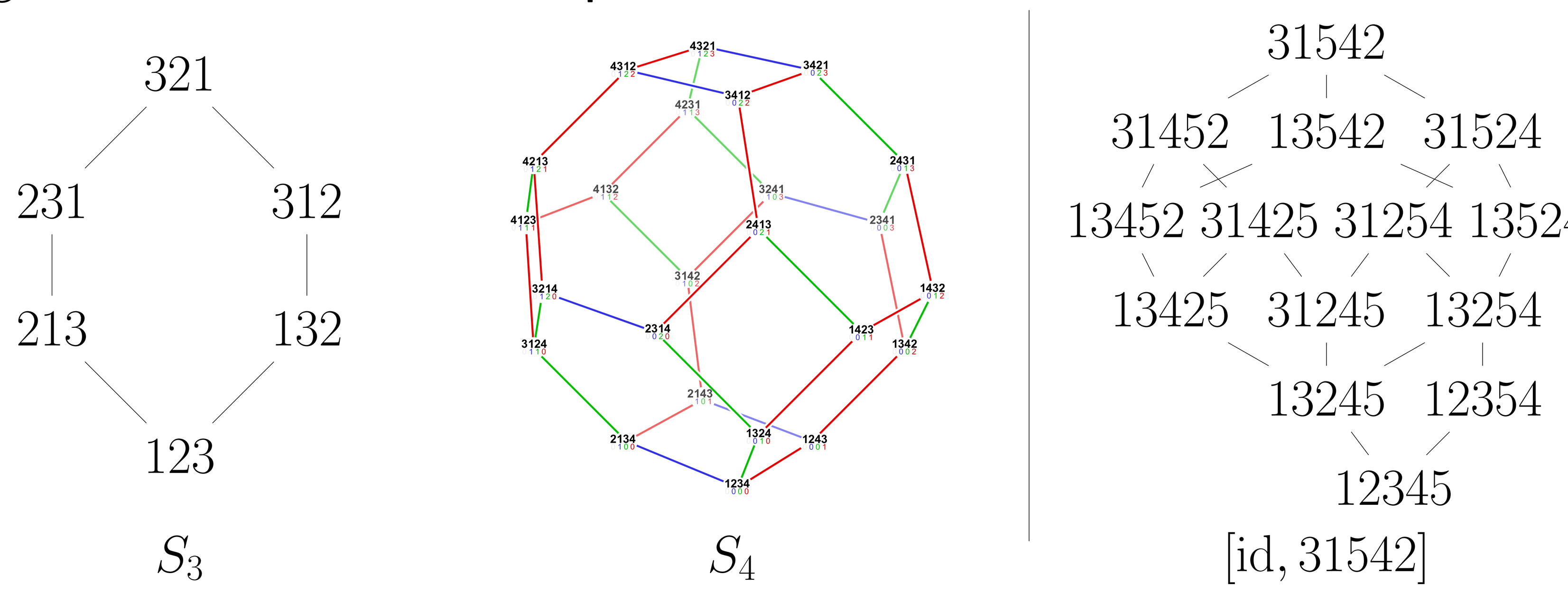
### Lemma (Chan-Haddadan-H.-Moci (CHHM), 2017)

The maxchain distribution $\max_{J(P)}$ is toggle-symmetric (so $J(P)$ tCDE $\Rightarrow$ CDE).

This connects the CDE property to **Dynamical Algebraic Combinatorics** and the study of the **rowmotion** operator on order ideals.

## Weak order on the symmetric group

**Weak order** is a well-known partial order on the symmetric group $S_n$, whose Hasse diagram is the 1-skeleton of the **permutohedron**:



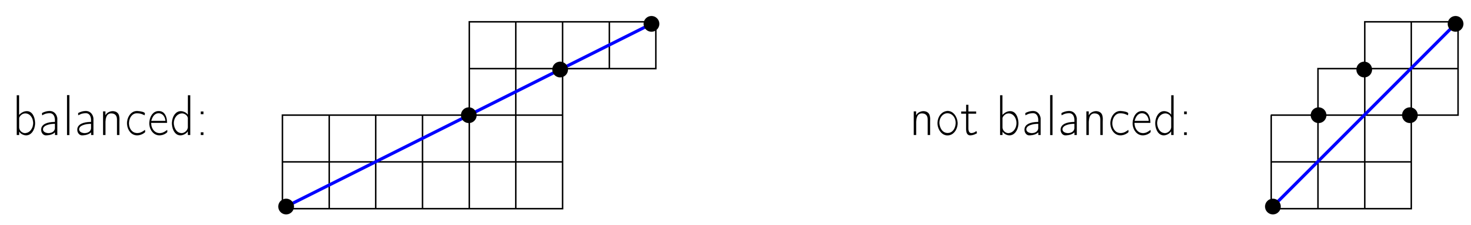

Maximal chains in initial intervals of weak order correspond to **reduced words** and are therefore significant in combinatorial Schubert calculus.

Weak order is not a distributive lattice, but it is a **semidistributive lattice**.

Recent work of Reading, Barnard, and Thomas-Williams gives a **semidistributive extension of toggling**, leading to notions of toggle-symmetric distributions, the tCDE property, et cetera, for semidistributive lattices.

## CDE intervals of Young's lattice & balanced shapes

Say a skew shape $\lambda/\nu$ is **balanced** if all its outward corners occur on main antidiagonal:




Balanced shapes include...

rectangles　staircases　rectangular staircases　and more

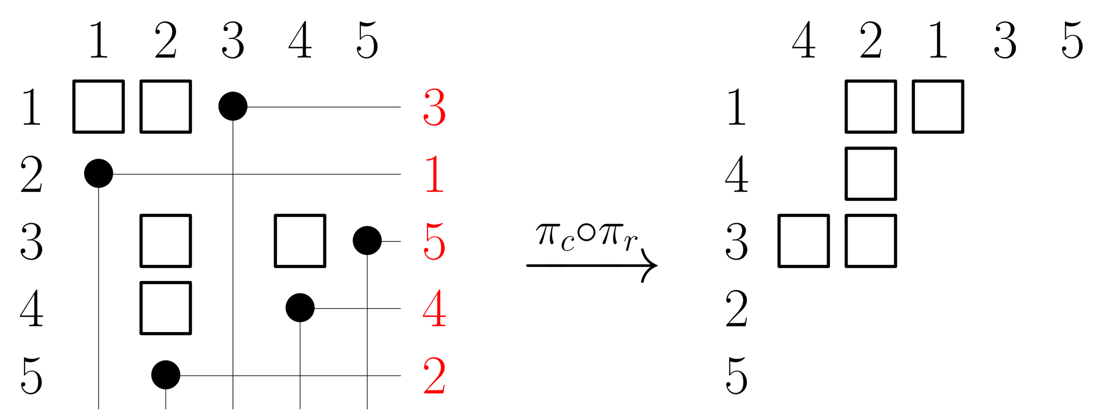
### Theorem (CHMM, 2017)

Let $\sigma = \lambda/\nu$ be a balanced shape of height $a$ and width $b$. Then the interval $[\nu, \lambda]$ of Young's lattice is tCDE with edge density $ab/(a+b)$.

As a corollary, we obtain a **homomesy** result for the action of rowmotion on these Young's lattice intervals.

## CDE intervals of weak order & vexillary permutations

A permutation $w \in S_n$ is **(skew) vexillary** of (skew) shape $\sigma$ if its **Rothe diagram** can be transformed to $\sigma$ by permuting rows and columns:




$w = 31542 \in S_5$ is skew vexillary of sh. $(3,2,2)/(1,1)$

### Theorem (H., 2019; conjectured by RTY)

Let $w \in S_n$ be a skew vexillary permutation of balanced shape $\sigma$. Then the initial interval $[\mathrm{id}, w]$ of weak order is tCDE (hence, CDE) with edge density $ab/(a+b)$, where $a$ and $b$ are the height and width of $\sigma$.

As a corollary, we obtain a homomesy result for a semidistributive generalization of rowmotion acting on these weak order intervals.

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