K-theoretic polynomials

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Background

One of the gems of 20th-century mathematics is the theory of symmetric functions and symmetric polynomials. Interpreting Schur functions through the cohomology of Grassmannians leads one to consider **K-theoretic** analogues of the classical bases. Additionally, we wish to lift the theory of symmetric polynomials to larger rings of **quasisymmetric** and **asymmetric** polynomials.

We introduce two new bases of $\mathrm{ASym}_n := \mathbb{Z}[\beta][x_1,\ldots,x_n]$. The **quasiLascoux** basis is a K-theoretic deformation of the quasikey basis that is also an asymmetric lift of quasiGrothendieck polynomials, a refinement of the Lascoux basis, and a simultaneous coarsening of the glide and Lascoux atom bases. **Kaons** are K-theoretic deformations of pions that are simultaneous refinements of glides and Lascoux atoms.

Three worlds

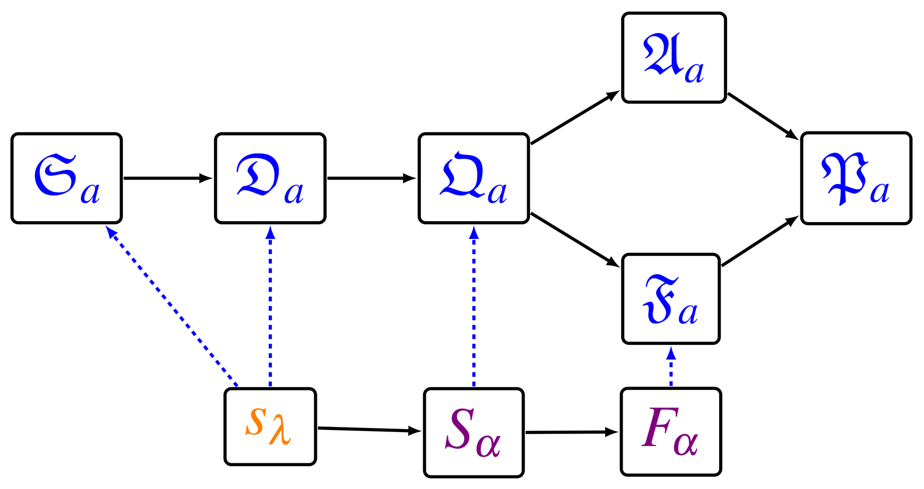
A polynomial $f \in \mathbb{Z}[\beta][x_1,\ldots,x_n]$ is **symmetric** if it is fixed by the action of S_n permuting subscripts.

We say f is **quasisymmetric** if the coefficient of $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k}$ equals the coefficient of $x_{j_1}^{\alpha_1}x_{j_2}^{\alpha_2}\cdots x_{j_k}^{\alpha_k}$ for every sequence $j_1 < j_2 < \cdots < j_k$.

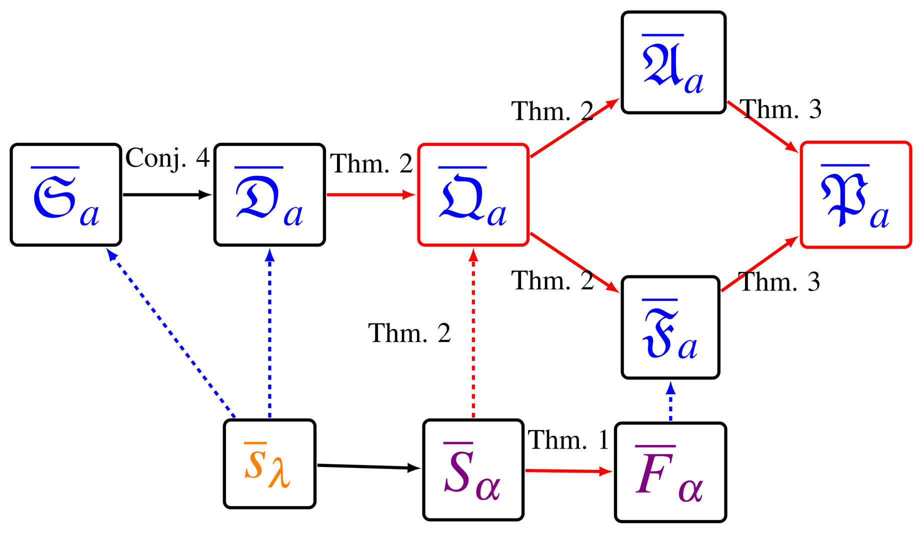
We call an arbitrary polynomial $f \in \mathbb{Z}[\beta][x_1,\ldots,x_n]$ **asymmetric**.

$$\mathrm{Sym}_n \subset \mathrm{QSym}_n \subset \mathrm{ASym}_n$$

Relations of cohomological bases



Relations of K-theory bases



K-theory deformations

Sym_n	Schur polynomial s_λ	symmetric Grothendieck polynomial $\overline{s}_\lambda$ [B02,MPSc18]
QSym_n	monomial quasisymmetric polynomial M_α	multimonomial polynomial $\overline{M}_\alpha$ [LP07]
	fundamental quasisymmetric polynomial F_α	multifundamental polynomial $\overline{F}_\alpha$ [LP07,Pat16,PS19]
	quasiSchur polynomial S_α	quasiGrothendieck polynomial $\overline{S}_\alpha$ [M16,MPSe18]
ASym_n	Schubert polynomial $\mathfrak{S}_a$	Grothendieck polynomial $\overline{\mathfrak{S}}_a$ [LS82,FK94,KM05]
	Demazure character/key polynomial $\mathfrak{D}_a$	Lascoux polynomial $\overline{\mathfrak{D}}_a$ [L01,RY15,K16,M16,MPSe18]
	quasikey polynomial $\mathfrak{Q}_a$	quasiLascoux polynomial $\overline{\mathfrak{Q}}_a$ [MPSe18]
	Demazure atom/standard basis $\mathfrak{A}_a$	Lascoux atom $\overline{\mathfrak{A}}_a$ [M16,MPSe18]
	pion/fundamental particle $\mathfrak{P}_a$	kaon $\overline{\mathfrak{P}}_a$ [MPSe18]
	fundamental slide polynomial $\mathfrak{F}_a$	glide polynomial $\overline{\mathfrak{F}}_a$ [PS19,MPSe18]

Kaons

A **weak komposition** is a weak composition whose positive integers are colored arbitrarily black or red. The **excess** $\mathrm{ex}(b)$ of a weak komposition b is its number of red entries. A weak komposition b is a **glide** of a if b can be obtained from a by the following local moves on the colored word:

(m.1) $0p \Rightarrow p0$, (for $p \in \mathbb{Z}_{>0}$);
(m.2) $0p \Rightarrow qr$ (for $p,q,r \in \mathbb{Z}_{>0}$ with $q+r=p$);
(m.3) $0p \Rightarrow q\textcolor{red}{r}$ (for $p,q,r \in \mathbb{Z}_{>0}$ with $q+r=p+1$).

Let a be a weak composition with nonzero entries in positions $n_1 < \cdots < n_\ell$. The weak komposition b is a **mesonic glide** of a if b can be obtained from a by a finite sequence of the local moves (m.1), (m.2), and (m.3) that never applies (m.1) at positions $n_j - 1$ and n_j for any j.

Definition 2: Let a be a weak composition. The **kaon** $\overline{\mathfrak{P}}_a$ is the following generating function for mesonic glides:

$$\overline{\mathfrak{P}}_a := \sum_b \beta^{\mathrm{ex}(b)}\mathbf{x}^b, \text{ where the sum is over all mesonic glides of } a.$$

Theorem 3: *The kaons $\{\overline{\mathfrak{P}}_a\}$ are a basis of ASym_n. They deform the fundamental particles, in that $\overline{\mathfrak{P}}_a$ recovers $\mathfrak{P}_a$ at $\beta = 0$. Kaons refine both glides and Lascoux atoms; that is,*

$$\overline{\mathfrak{F}}_a = \sum_b P_b^a\,\overline{\mathfrak{P}}_b \quad \text{and} \quad \overline{\mathfrak{A}}_a = \sum_b Q_b^a\,\overline{\mathfrak{P}}_b, \quad \text{where } P_b^a, Q_b^a \in \mathbb{N}[b].$$

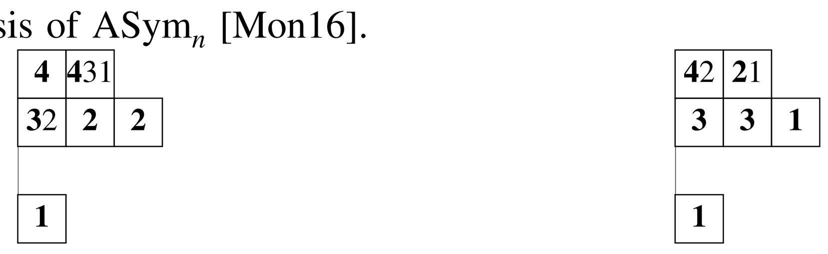
QuasiLascoux polynomials

The **skyline diagram** $D(a)$ of a weak composition a has a_i left-justified boxes in row i (for us, row 1 is the lowest). A **triple** of $D(a)$ is a set of 3 boxes with 2 adjacent in a row and either the third box above the right box and the lower row weakly longer, or the third box below the left box and the higher row strictly longer. Given a numerical filling of the skyline diagram, a triple is called a **coinversion triple** if $\alpha \leq \gamma \leq \beta$ (where γ is the label of the third box); otherwise, it is an **inversion triple**. A **set-valued filling** of a skyline diagram is an assignment of a non-empty set of positive integers to each box. The greatest entry in each box is the **anchor**; other entries are **free**. A set-valued filling is **semistandard** if (S.1) entries do not repeat in a column, (S.2) rows are weakly decreasing (where sets $A \geq B$ if $\min A \geq \max B$), (S.3) every triple of anchors is an inversion triple, (S.4) each free entry appears with the least anchor in its column such that (S.2) is not violated, and (S.5) anchors in column 1 equal their row indices.

Given a set-valued filling F of shape a, the **weight** of F is the weak composition $\mathrm{wt}(F) = (c_1,\ldots,c_n)$ where c_i is the number of i's in F. The **excess** $\mathrm{ex}(F)$ of F is its number of free entries. For a weak composition a, let $\overline{\mathfrak{A}}\mathrm{SSF}(a)$ be the set of semistandard set-valued skyline diagrams of shape a. Then, the **Lascoux atom** $\overline{\mathfrak{A}}_a$ is

$$\overline{\mathfrak{A}}_a = \sum_{F \in \overline{\mathfrak{A}}\mathrm{SSF}(a)} \beta^{\mathrm{ex}(F)}\mathbf{x}^{\mathrm{wt}(F)}.$$

Lascoux atoms form a basis of ASym_n [Mon16].



Definition 1: For a weak composition a, the **quasiLascoux polynomial** $\overline{\mathfrak{Q}}_a$ is

$$\overline{\mathfrak{Q}}_a = \sum_{\substack{b \geq a \\ b^+ = a^+}} \overline{\mathfrak{A}}_b.$$

Theorem 1: *Each quasiGrothendieck polynomial [Mon16] $\overline{S}_\alpha \in \mathrm{QSym}_n$ is a positive sum of multifundamental quasisymmetric polynomials [LP07]. That is,*

$$\overline{S}_\alpha = \sum_\gamma J_\gamma^\alpha\,\overline{F}_\gamma, \text{ where } J_\gamma^\alpha \in \mathbb{N}[\beta].$$

Theorem 2: *The quasiLascoux polynomials $\{\overline{\mathfrak{Q}}_a\}$ are a basis of ASym_n. They lift the quasiGrothendieck basis in that $\{\overline{\mathfrak{Q}}_a\} \cap \mathrm{QSym}_n = \{\overline{S}_\alpha\}$. Moreover, they deform the quasikeys, in that $\overline{\mathfrak{Q}}_a$ recovers $\mathfrak{Q}_a$ at $\beta = 0$. Finally, the quasiLascoux polynomials refine Lascoux polynomials and are refined by both glides and by Lascoux atoms; that is,*

$$\overline{\mathfrak{D}}_a = \sum_b L_b^a\,\overline{\mathfrak{Q}}_b, \quad \overline{\mathfrak{Q}}_a = \sum_b M_b^a\,\overline{\mathfrak{F}}_b, \quad \text{and} \quad \overline{\mathfrak{Q}}_a = \sum_b N_b^a\,\overline{\mathfrak{A}}_b, \quad \text{where } L_b^a, M_b^a, N_b^a \in \mathbb{N}[\beta].$$

Conjectures

Conjecture 1: *For a weak composition a, $\sum_b M_b^a(-1) \in \{0,1\}$ and $\sum_b Q_b^a(-1) \in \{0,1\}$, where both sums are over all weak compositions b.*

Conjecture 2: *For a,b weak compositions, $\overline{\mathfrak{D}}_a \cdot \overline{\mathfrak{D}}_b$ is a positive sum of Lascoux atoms.*

Conjecture 3: *For any weak compositions a and b, the product $\overline{\mathfrak{P}}_a \cdot \overline{\mathfrak{F}}_b$ of a kaon and a glide polynomial expands positively in kaons.*

Conjecture 4: (Reiner, Shimozono, Yong) *Each Grothendieck polynomial $\overline{\mathfrak{S}}_a$ is a positive sum of Lascoux polynomials $\overline{\mathfrak{D}}_b$.*

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