Skew Hook Formula for d-Complete Posets via Equivariant K-Theory

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Main Result

The notion of d-complete posets was introduced by Proctor as a generalization of Young diagrams $D(\lambda)$, shifted Young diagrams $S(\mu)$, and rooted trees.

	skew hook formula		hook formula
P-partitions	Main Theorem	$\xrightarrow{F=\emptyset}$	Peterson–Proctor
	$\downarrow P\setminus F = D(\lambda/\mu)$, $z_i = q$		$\downarrow P = D(\lambda)$, $z_i = q$
reverse plane partitions	Morales–Pak–Panova	$\xrightarrow{\mu=\emptyset}$	Stanley
	$\downarrow$ "$q \to 1$"		$\downarrow$ "$q \to 1$"
standard tableaux	Naruse	$\xrightarrow{\mu=\emptyset}$	Frame–Robinson–Thrall

For a poset P and an order filter F of P, we put

$$\mathcal{A}(P\setminus F) = \left\{ \begin{array}{r} \text{order-reversing maps } \sigma : P\setminus F \to \mathbb{N} \\ \text{(i.e., } (P\setminus F)\text{-partitions)} \end{array} \right\}.$$

Main Theorem

Let

P: a d-complete poset $\supset F$: an order filter,

and $c : P \to I$ a d-complete coloring. Then the multivariate generating function of $(P\setminus F)$-partitions w.r.t. indeterminates $\boldsymbol{z} = (z_i)_{i\in I}$ is given by

$$\sum_{\sigma\in\mathcal{A}(P\setminus F)} \boldsymbol{z}^\sigma = \sum_{D\in\mathcal{E}_P(F)} \frac{\prod_{u\in B(D)} \boldsymbol{z}[H_P(u)]}{\prod_{v\in P\setminus D}(1-\boldsymbol{z}[H_P(v)])},$$

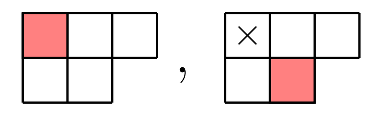
where

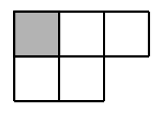
$$\boldsymbol{z}^\sigma = \prod_{v\in P} (z_{c(v)})^{\sigma(v)},$$

$\mathcal{E}_P(F) = \{$excited diagrams of F in $P\}$,
$B(D) =$ the set of excited peaks for D,
$\boldsymbol{z}[H_P(v)] =$ the hook monomial at $v \in P$.

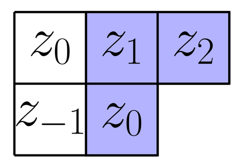
See below for definitions.

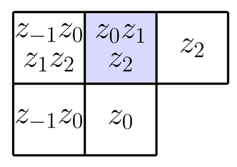
Example. If $P = D(3,2)$ and $F = D(1)$ are Young diagrams, then

$\mathcal{E}_P(F) = \{$  $\}$,

where ■ is a cell in an excited diagram and ☒ is an excited peak. Hence the generating function of reverse plane partition of skew shape $(3,2)/(1) =$  is given by

$$\sum_{\sigma\in\mathcal{A}(D((3,2)/(1)))} \boldsymbol{z}^\sigma = \frac{1}{(1-z_0z_1z_2)(1-z_2)(1-z_{-1}z_0)(1-z_0)} + \frac{z_{-1}z_0z_1z_2}{(1-z_{-1}z_0z_1z_2)(1-z_0z_1z_2)(1-z_2)(1-z_{-1}z_0)}.$$

$(z_{c(v)})_v$: 

$(\boldsymbol{z}[H_P(v)])_v$: 

Combinatorial Definitions for d-Complete Posets

d-Complete Posets

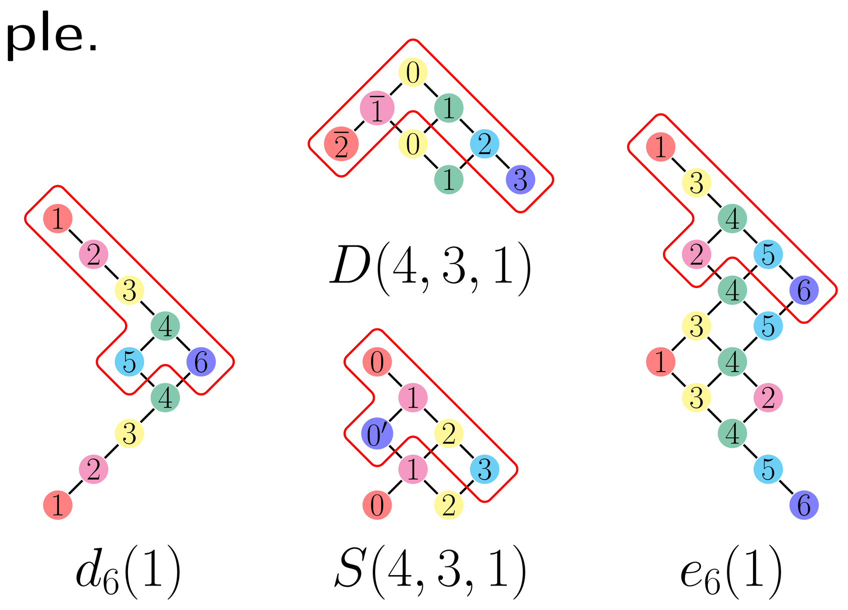
d-Complete posets are defined by certain local structural conditions (omitted here) in terms of d_k-intervals.

- An interval of a poset P is called a **d_k-interval** if it is isomorphic to the $(2k-2)$-element poset $d_k(1) = [k-2] \oplus ([1]+[1]) \oplus [k-2]$.

Let P be a connected d-complete poset.

- The **top tree** Γ of P is the subgraph of the Hasse diagram of P given by
$$\Gamma = \{x \in P : [x, v_{\max}] \text{ is a chain}\},$$
where $v_{\max}$ is the unique maximal element of P.
- A **d-complete coloring** of P is a map $c : P \to I$ such that $c|_\Gamma : \Gamma \to I$ is a bijection and
$$\text{if } [v,u] \text{ is a } d_k\text{-interval, then } c(u) = c(v).$$

Example.

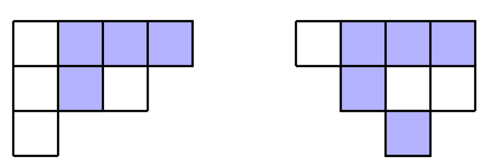


Hook Monomials

Let P be a connected d-complete poset with d-complete coloring $c : P \to I$. Let $\boldsymbol{z} = (z_i)_{i\in I}$ be indeterminates. The **hook monomials** $\boldsymbol{z}[H_P(u)]$ $(u \in P)$ are defined inductively (from bottom to top) as follows:

- If u is not the top of any d_k-interval, then
$$\boldsymbol{z}[H_P(u)] = \prod_{w\le u} z_{c(w)}.$$
- If u is the top of a d_k-interval $[v,u]$, then
$$\boldsymbol{z}[H_P(u)] = \frac{\boldsymbol{z}[H_P(x)]\cdot\boldsymbol{z}[H_P(y)]}{\boldsymbol{z}[H_P(v)]},$$
where x and y are the incomparable elements in $[v,u]$.

Example. For Young diagrams $D(\lambda)$ and shifted Young diagrams $S(\mu)$, we have the classical notion of hooks $H_{D(\lambda)}(u) \subset D(\lambda)$ and shifted hooks $H_{S(\mu)}(u) \subset S(\mu)$:



Then we have

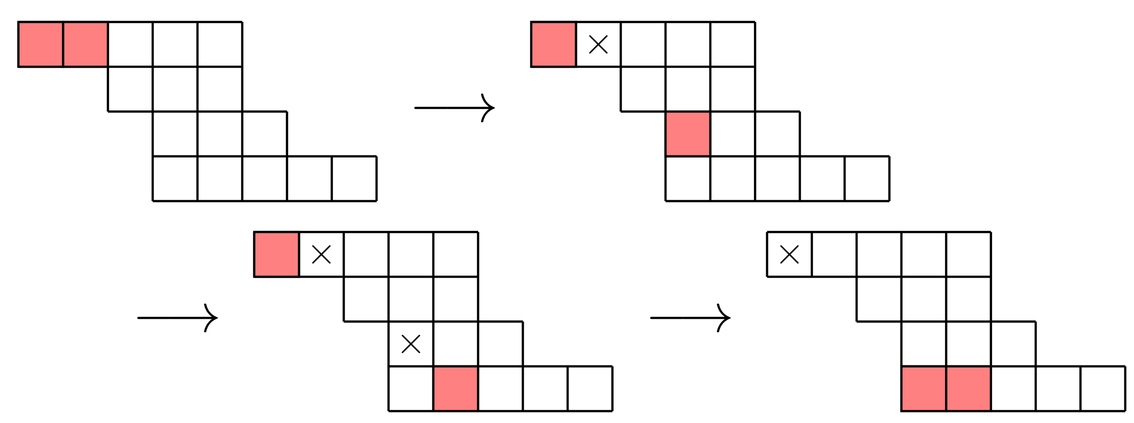
$$\boldsymbol{z}[H_P(u)] = \prod_{x\in H_P(u)} z_{c(x)}$$

for $P = D(\lambda)$ or $S(\mu)$.

Excited Diagrams and Excited Peaks

- Let D be a subset of a d-complete poset and $u \in D$. When there is a d_k-interval $[v,u]$ such that $v \notin D$ and
$$\text{if } x \lessdot u \text{ or } x \gtrdot v, \text{ then } x \notin D,$$
we define an **elementary excitation** of D by
$$\alpha_u(D) = D \setminus \{u\} \cup \{v\}.$$
- An **excited diagram** of F in P is a subset of P obtained from F after a sequence of elementary excitations $D \to \alpha_u(D)$.
- For an excited diagram D, the set $B(D)$ of **excited peaks** is defined recursively by $B(F) = \emptyset$ and
$$B(\alpha_u(D)) = B(D) \setminus \{x \in P : x \lessdot u \text{ or } x \gtrdot v\} \cup \{u\}.$$

Example. If $P = e_6(1)$ and F is the two-element order filter of P, then there are four excited diagrams of F in P:



Proof Idea of Main Theorem (via Equivariant K-Theory)

Let P be a connected d-complete poset with maximum element $v_{\max}$, top tree Γ and d-complete coloring $c : P \to I$. By regarding Γ as a (simply-laced) Dynkin diagram, we associate

- $W = \langle s_i : i \in I\rangle$: the Kac–Moody Weyl group,
- W^J : the parabolic quotient corresp. to J,
- $\Lambda \supset \{\alpha_i\}_{i\in I}$: the weight lattice and the simple roots,
- $\mathcal{G} \supset \mathcal{T}$: the Kac–Moody group and a maximal torus,
- $\mathcal{P}_J^-$: the max. parabolic subgroup of $\mathcal{G}$ corresp. to J,
- $\mathcal{X} = \mathcal{G}/\mathcal{P}_J^-$: the thick partial flag variety,

where $J = I \setminus \{c(v_{\max})\}$.

We fix a linear extension $(p_1, \dots, p_N)$ of P. For a subset $E = \{p_{i_1}, \dots, p_{i_r}\} \subset P$ $(i_1 < \dots < i_r)$, we define

$$w_E = s_{c(p_{i_1})} \cdots s_{c(p_{i_r})}.$$

Let $K_{\mathcal{T}}(\mathcal{X})$ be the $\mathcal{T}$-equivariant K-theory of $\mathcal{X}$. For $v \in W^J$, let $[\mathcal{O}_v]$ be the class of the structure sheaf of the Schubert variety $\mathcal{X}_v \subset \mathcal{X}$. For $w \in W^J$, the inclusion map $\iota_w : \{w\mathcal{P}_J^-/\mathcal{P}_J^-\} \to \mathcal{X}$ induces the localization map $\iota_w^* : K_{\mathcal{T}}(\mathcal{X}) \to K_{\mathcal{T}}(\mathrm{pt}) \cong \mathbb{Z}[\Lambda]$. We put

$$\xi^v|_w = \iota_w^*([\mathcal{O}_v]).$$

Main Theorem follows from

Key Identities

Under the identification $z_i = e^{\alpha_i}$ $(i \in I)$, we have

$$\sum_{\sigma\in\mathcal{A}(P\setminus F)} \boldsymbol{z}^\sigma = \frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}},$$

$$\sum_{D\in\mathcal{E}_P(F)} \frac{\prod_{u\in B(D)} \boldsymbol{z}[H_P(u)]}{\prod_{v\in P\setminus D}(1-\boldsymbol{z}[H_P(v)])} = \frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}}.$$

The full paper is available at arXiv:1802.09748 and will appear in Algebraic Combinatorics.

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