Homomorphisms on Noncommutative Symmetric Functions and Permutation Enumeration

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Abstract

We present a unifying framework for deriving permutation enumeration formulas involving restrictions on descent compositions or permutation statistics describable in terms of descent compositions: these formulas can be proven by first deriving a lifting of the formula in the algebra of noncommutative symmetric functions and then applying an appropriate homomorphism. We give several applications of this method including rederivations and extensions of classical results in the literature as well as new results.

Background

- For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, we call i a *descent* of π if $\pi_i > \pi_{i+1}$.
- Every permutation can be uniquely decomposed into a sequence of maximal increasing consecutive subsequences, called *increasing runs*.
- The *descent composition* of π is the integer composition whose parts are the increasing run lengths of π.
- Let $\mathrm{Comp}(\pi)$ denote the descent composition of π.

A Classical Formula

- David and Barton [1] proved that

$$\left[\sum_{n=0}^{\infty}\left(\frac{x^{mn}}{(mn)!}-\frac{x^{mn+1}}{(mn+1)!}\right)\right]^{-1}$$

is the exponential generating function for permutations in which every increasing run has length less than m.

Noncommutative Symmetric Functions

- For a composition $L = (L_1,\dots,L_k)$, let

$$\mathbf{r}_L := \sum_{i_1,\dots,i_n} X_{i_1}X_{i_2}\cdots X_{i_n}$$

where the sum is over all $i_1,\dots,i_n$ satisfying

$$\underbrace{i_1 \le \cdots \le i_{L_1}}_{L_1} > \underbrace{i_{L_1+1} \le \cdots \le i_{L_1+L_2}}_{L_2} > \cdots > \underbrace{i_{L_1+\cdots+L_{k-1}+1} \le \cdots \le i_n}_{L_k}.$$

- The algebra **Sym** of *noncommutative symmetric functions* is the $\mathbb{Q}$-subalgebra of $\mathbb{Q}\langle\langle X_1, X_2, \dots\rangle\rangle$ with basis $\{\mathbf{r}_L\}$.
- Define $\mathbf{h}_n := \mathbf{r}_{(n)}$ for $n \ge 1$ and $\mathbf{h}_0 := 1$. Then the $\mathbf{h}_n$ are algebraically independent and generate **Sym**.

The Homomorphism Φ

- Define $\Phi\colon \mathbf{Sym} \to \mathbb{Q}[[x]]$ by $\Phi(\mathbf{h}_n) := x^n/n!$.
- Then $\Phi(\mathbf{r}_L) = \beta(L)x^n/n!$ where $\beta(L)$ is the number of permutations with descent composition L.

The Homomorphism $\hat{\Phi}$

- We call i an *alternating descent* of $\pi = \pi_1\pi_2\cdots\pi_n$ if i is odd and $\pi_i > \pi_{i+1}$ or if i is even and $\pi_i < \pi_{i+1}$.
- An *alternating run* of π is a maximal increasing subsequence containing no alternating descents, and define the *alternating descent composition* analogously.
- Define $\hat{\Phi}\colon \mathbf{Sym} \to \mathbb{Q}[[x]]$ by $\hat{\Phi}(\mathbf{h}_n) := E_n x^n/n!$ where E_n is the nth Euler number.
- Then $\hat{\Phi}(\mathbf{r}_L) = \hat{\beta}(L)x^n/n!$ where $\hat{\beta}(L)$ is the number of permutations with alternating descent composition L.

The Homomorphism Φ_q

- An *inversion* of $\pi = \pi_1\pi_2\cdots\pi_n$ is a pair of indices (i,j) with $1 \le i < j \le n$ such that $\pi_i > \pi_j$.
- Let $\mathrm{inv}(\pi)$ denote the number of inversions of π.
- Define $\Phi_q\colon \mathbf{Sym} \to \mathbb{Q}[[q,x]]$ by $\Phi_q(\mathbf{h}_n) := x^n/[n]_q!$.
- Then $\Phi_q(\mathbf{r}_L) = \beta_q(L)x^n/[n]_q!$ where $\beta_q(L) := \sum_{\mathrm{Comp}(\pi)=L} q^{\mathrm{inv}(\pi)}$.

The Homomorphism Φ_{des}

- The *Hadamard product* $*$ on formal power series in t is given by

$$\left(\sum_{n=0}^{\infty} a_n t^n\right) * \left(\sum_{n=0}^{\infty} b_n t^n\right) = \sum_{n=0}^{\infty} a_n b_n t^n.$$

- Let $\mathbb{Q}[[t*,x]]$ denote the $\mathbb{Q}$-algebra of formal power series in t and x, where the multiplication is Hadamard product in t.
- Let $\mathrm{ides}(\pi)$ denote the number of descents of π^{-1}.
- Define $\Phi_{\mathrm{des}}\colon \mathbf{Sym} \to \mathbb{Q}[[t^*,x]]$ by $\Phi_{\mathrm{des}}(\mathbf{h}_n) = tx^n/(1-t)^{n+1}$.
- Then $\Phi_{\mathrm{des}}(\mathbf{r}_L) = \sum_{\mathrm{Comp}(\pi)=L} t^{\mathrm{ides}(\pi)+1}x^n/(1-t)^{n+1}$.

An Example: Variations of David and Barton's Formula

- **Lemma:** Let m be a positive integer. Then

$$\sum_L \mathbf{r}_L = \left(\sum_{n=0}^{\infty}(\mathbf{h}_{mn} - \mathbf{h}_{mn+1})\right)^{-1} \qquad (\heartsuit)$$

where the left sum is over all compositions L with all parts less than m.
- Applying Φ to $(\heartsuit)$ yields David and Barton's formula.
- Applying $\hat{\Phi}$ to $(\heartsuit)$ yields the exponential generating function

$$\left[\sum_{n=0}^{\infty}\left(E_{mn}\frac{x^{mn}}{(mn)!}-E_{mn+1}\frac{x^{mn+1}}{(mn+1)!}\right)\right]^{-1}$$

for permutations in which every alternating run has length less than m.
- Let $\mathrm{Av}_n(\underline{12\cdots m})$ denote the set of permutations of length n in which every increasing run has length less than m. Applying Φ_q to $(\heartsuit)$ yields

$$\sum_{n=0}^{\infty}\sum_{\pi\in\mathrm{Av}_n(\underline{12\cdots m})} q^{\mathrm{inv}(\pi)}\frac{x^n}{[n]_q!} = \left[\sum_{n=0}^{\infty}\left(\frac{x^{mn}}{[mn]_q!}-\frac{x^{mn+1}}{[mn+1]_q!}\right)\right]^{-1}.$$

- Let $M_{m,n}^{\mathrm{ides}}(t) := \sum_{\pi\in\mathrm{Av}_n(\underline{12\cdots m})} t^{\mathrm{ides}(\pi)+1}$. Applying Φ_{des} to $(\heartsuit)$ yields

$$\sum_{n=0}^{\infty}\frac{M_{m,n}^{\mathrm{ides}}(t)}{(1-t)^{n+1}}x^n = \sum_{k=0}^{\infty}\left[\sum_{j=0}^{\infty}\left(\binom{k+jm-1}{k-1}x^{jm} - \binom{k+jm}{k-1}x^{jm+1}\right)\right]^{-1} t^k.$$

Other Applications

- See [2, 3, 4, 5] for other applications of these homomorphisms to permutation enumeration; these include many new formulas as well as rederivations of formulas by Carlitz, Chebikin, Elizalde–Noy, Petersen, Remmel, Stanley, and Stembridge.

References

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