The lattice permutation condition for Kronecker tableaux

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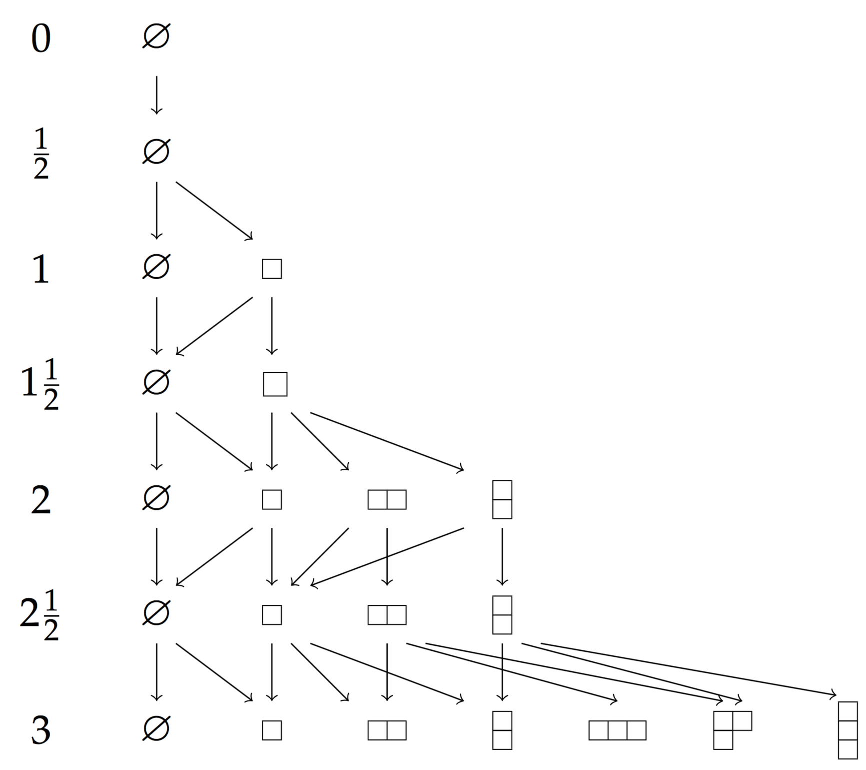
From Young tableaux to Kronecker tableaux

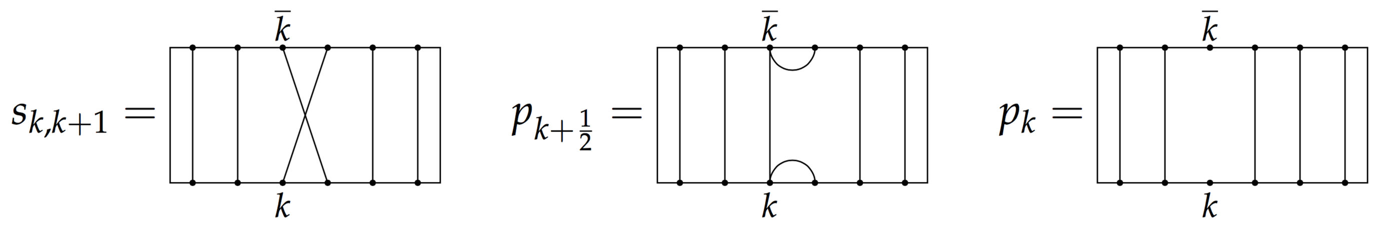
We calculate *stable* Kronecker coefficients in terms of Kronecker tableaux. Kronecker tableaux are the paths in the graph to the right and they provide bases of "skew Specht modules" for the partition algebra, $P_s(n)$. A momentary glance at the graph reveals a very familiar subgraph: namely Young's graph (with each level doubled up). The stable Kronecker coefficients labelled by triples from this subgraph are well-understood — the values of these coefficients can be calculated via the Littlewood–Richardson rule. This rule counts the tableaux satisfying the semistandard and lattice permutation conditions. We generalise the lattice permutation condition from the Young subgraph to all Kronecker tableaux.

This results in an algorithm for counting a large new class of stable Kronecker coefficients (subsuming and unifying the Littlewood–Richardson and two 2-part partition cases). Most promisingly, this result counts explicit homomorphisms and thus works on a structural level above any description of a family of Kronecker coefficients since those first considered by Littlewood–Richardson.



The partition algebra action on standard Kronecker tableaux

The partition algebra is generated as an algebra by the diagrams pictured below modulo a long list of relations. One can visualise any product in this algebra as simply being given by concatenation of diagrams, modulo some surgery to remove closed loops:



Given partitions λ and ν, define the *skew Specht module*

$$\Delta_s(\nu \setminus \lambda) = \mathrm{Span}\{\mathsf{t} \mid \mathsf{t} \in \mathrm{Std}_s(\nu \setminus \lambda)\}$$

where $\mathrm{Std}_s(\nu \setminus \lambda)$ is all paths of length s from λ to ν. For $n \geq 2s$, the stable Kronecker coefficients are the multiplicities

$$\overline{g}(\lambda, \nu, \mu) = \dim_{\mathbb{C}}(\mathrm{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s(\nu \setminus \lambda))).$$

Fix $\mathsf{t} \in \mathrm{Std}_r(\nu)$ and $1 \leq k \leq r$ and suppose that

$$\mathsf{t}(k-1) \xrightarrow{-t} \mathsf{t}(k-\tfrac{1}{2}) \xrightarrow{+u} \mathsf{t}(k+1) \xrightarrow{-v} \mathsf{t}(k+\tfrac{1}{2}) \xrightarrow{+w} \mathsf{t}(k+1).$$

We define $\mathsf{t}_{k\leftrightarrow k+1} \in \mathrm{Std}_r(\nu)$ to be the tableau, if it exists, determined by $\mathsf{t}_{k\leftrightarrow k+1}(l) = \mathsf{t}(l)$ for $l \neq k, k \pm \frac{1}{2}$ and

$$\mathsf{t}_{k\leftrightarrow k+1}(k-1) \xrightarrow{-v} \mathsf{t}_{k\leftrightarrow k+1}(k-\tfrac{1}{2}) \xrightarrow{+w} \mathsf{t}_{k\leftrightarrow k+1}(k) \xrightarrow{-t} \mathsf{t}_{k\leftrightarrow k+1}(k+\tfrac{1}{2}) \xrightarrow{+u} \mathsf{t}_{k\leftrightarrow k+1}(k+1).$$

Theorem

The action of the Coxeter generators on paths can is as follows:

$$\mathsf{t} \cdot s_{k,k+1} = \mathsf{t}_{k\leftrightarrow k+1} + \ldots \text{ if } \mathsf{t}_{k\leftrightarrow k+1} \text{ exists}$$

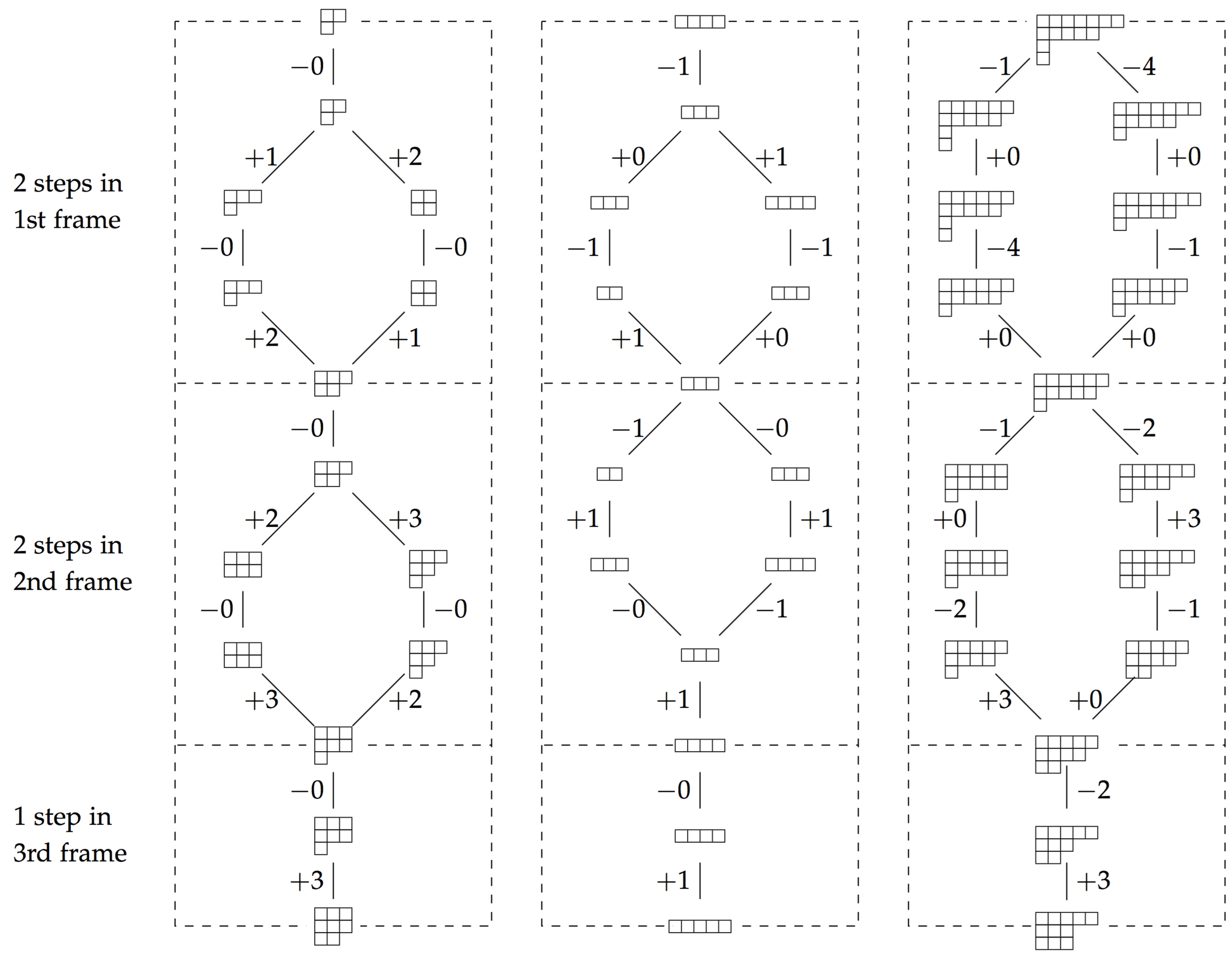
for $1 \leq k < s$ where $\ldots$ denotes more dominant terms (which we can control).

For triples of maximal depth (those with $s = |\nu| - |\lambda|$) we have

$$\overline{g}(\lambda, \nu, \mu) = \dim_{\mathbb{C}}(\mathrm{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s(\nu \setminus \lambda))) = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathfrak{S}_s}(S(\mu), S(\nu \setminus \lambda))) = c(\lambda, \nu, \mu)$$

where $c(\lambda, \nu, \mu)$ is the classical Littlewood–Richardson coefficient.

Semistandard Kronecker tableaux



Definition

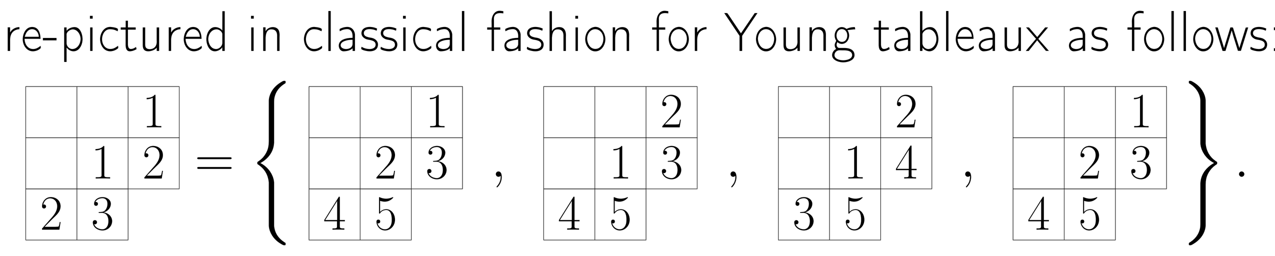
Let (λ, ν, s) and $\mu = (\mu_1, \ldots, \mu_\ell) \vdash s$, let $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_s(\nu \setminus \lambda)$.

- For $1 \leq k < s$ we write $\mathsf{s} \overset{k}{\sim} \mathsf{t}$ if $\mathsf{s} = \mathsf{t}_{k\leftrightarrow k+1}$.
- We write $\mathsf{s} \overset{\mu}{\sim} \mathsf{t}$ if there exists a sequence of standard Kronecker tableaux $\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_d \in \mathrm{Std}_s(\nu \setminus \lambda)$ such that
$$\mathsf{s} = \mathsf{t}_1 \overset{k_1}{\sim} \mathsf{t}_2,\ \mathsf{t}_2 \overset{k_2}{\sim} \mathsf{t}_3,\ \ldots\ , \mathsf{t}_{d-1} \overset{k_{d-1}}{\sim} \mathsf{t}_d = \mathsf{t}$$
for some $k_1, \ldots, k_{d-1} \in \{1, \ldots, s-1\} \setminus \{\sum_{i \leq c} \mu_i \mid c = 1, \ldots, l-1\}$. We define a tableau of weight μ to be an equivalence class of tableau under $\overset{\mu}{\sim}$, denoted $[\mathsf{t}]_\mu = \{\mathsf{s} \in \mathrm{Std}_s(\nu \setminus \lambda) \mid \mathsf{s} \overset{\mu}{\sim} \mathsf{t}\}$.
- A Kronecker tableau, $[\mathsf{t}]_\mu$, of shape $\nu \setminus \lambda$ and weight μ is semistandard if for any $\mathsf{s} \in [\mathsf{t}]_\mu$ and any $k \notin \{[\mu_c] \mid 1 \leq c < \ell\}$ the tableau $\mathsf{s}_{k\leftrightarrow k+1}$ exists. We let $\mathrm{SStd}(\nu \setminus \lambda, \mu)$ denote the set of semistandard Kronecker tableaux of shape $\nu \setminus \lambda$ and weight μ.

We record the weight of the semistandard Kronecker tableau as a series of frames on the branching graph. We then picture all the Kronecker tableaux in the orbit, as on the left.

The lattice permutation condition for Kronecker tableaux

The leftmost semistandard Kronecker tableau pictured above is of shape $(3,3,2) \setminus (2,1)$ and weight $\mu = (2,2,1)$ and is an orbit of 4 Kronecker tableaux. This can be re-pictured in classical fashion for Young tableaux as follows:



In order to define the lattice permutation condition, we need an ordering on steps

Definition

We order the "steps" in the branching graph of $P_s(n)$ as follows,

$$\underset{\text{move-up}}{(-\varepsilon_p, +\varepsilon_q)} < \underset{\text{dummy}}{(-\varepsilon_t, +\varepsilon_t)} < \underset{\text{move-down}}{(-\varepsilon_u, +\varepsilon_v)}$$

for $p > q$ and $u < v$. We refine this to a total order as follows,

- we order $(-\varepsilon_p, +\varepsilon_q) < (-\varepsilon_{p'}, +\varepsilon_{q'})$ if $q < q'$ or $q = q'$ and $p > p'$;
- we order $(-\varepsilon_t, +\varepsilon_t) < (-\varepsilon_{t'}, +\varepsilon_{t'})$ if $t > t'$;
- we order $(-\varepsilon_u, +\varepsilon_v) < (-\varepsilon_{u'}, +\varepsilon_{v'})$ if $u > u'$ or $u = u'$ and $v < v'$.

Definition

We encode the integral steps of $\mathsf{S} \in \mathrm{SStd}(\lambda, \mu)$ and their frames in a $2 \times s$ array, called the reverse reading word of S, as follows. The first row contains all the steps of any $\mathsf{s} \in \mathsf{S}$; the second row contains their corresponding frames. We order the columns increasingly using the ordering on steps. For two equal steps, we order the columns so that the frame numbers are weakly decreasing.

Loosely speaking, a co-Pieri triple (λ, ν, s) is any triple such that all tableaux of weight $\mu \vdash s$ are semistandard. For the Young subgraph, this is equivalent to $\nu \setminus \lambda$ having no two nodes in the same column.

Theorem

Let (λ, ν, μ) be a co-Pieri triple or a triple of maximal depth. Then the stable Kronecker coefficient $\overline{g}(\lambda, \nu, \mu)$ is given by the number of semistandard Kronecker tableaux of shape $\nu \setminus \lambda$ and weight μ whose reverse reading word is a lattice permutation.

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