A combinatorial interpretation of the number $h_d^*(\Delta_{k,n})$

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Introduction

- **Ehrhart polynomial, Ehrhart series, and $\mathbf{h}^*$-vector**
  -For an n-dimensional integral polytope $\mathcal{P} \subset \mathbb{R}^N$, the map $r \in \mathbb{Z} \to |r\mathcal{P} \cap \mathbb{Z}^N|$ is a polynomial function in r of degree n (Ehrhart, 1960s). We call this polynomial an Ehrhart polynomial of $\mathcal{P}$. Now Ehrhart series is defined as $\sum_{r=0}^{\infty} |r\mathcal{P} \cap \mathbb{Z}^N| t^r$. This infinite sum can be represented as a fraction of the form $\frac{h^*(t)}{(1-t)^{n+1}}$, where $h^*(t)$ is a polynomial of degree at most n. And h^*-vector of $\mathcal{P}$ is defined to be a coefficient vector of $h^*(t)$. We denote $h_i^*(\mathcal{P}) = [t^i]h^*(t)$.
- **About $\mathbf{h}^*$-vector...**
  -Sum of h^*-vector of $\mathcal{P}$ is a normalized volume of $\mathcal{P}$.
  -In general, the entries of the h^*-vector of an integral polytope are nonnegative integers ([4]).
- **Hypersimplex $\mathbf{\Delta_{k,n}}$**
  -The (k,n)-th hypersimplex is defined to be
  $$\Delta_{k,n} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid 0 \le x_i \le 1, x_1 + \cdots + x_n = k\}.$$
  The hypersimplex can be found in several algebraic and geometric contexts, for example, as a moment polytope for the torus action on the Grassmannian, or as a weight polytope for the fundamental representation of GL_n.
  -Normalized volume of the hypersimplex is an Eulerian number $A_{k,n-1}$, which counts number of $w \in S_{n-1}$ with $des(w) = k-1$.
- **Main question and previous results**
  -We have $\sum_{d=0}^{n-1} h_d^*(\Delta_{k,n}) = A_{k,n-1}$. Combinatorial number on the right-hand side splits into nonnegative integers on the left-hand side. It is natural to ask for the <u>combinatorial interpretation of the number $h_d^*(\Delta_{k,n})$.</u>
  -A combinatorial interpretation of $h_d^*(\Delta'_{k,n})$ is known, where $\Delta'_{k,n}$ is the hypersimplex with the lowest facet removed ([2]). Several statistics (descents, excedances, and covers) on permutations were used to give an interpretation.
  -In 2005, Katzman computed Hilbert series of algebras of Veronese type ([1]) which gave a formula for $h_d^*(\Delta_{k,n})$ as a special case. It is given as
  $$h_d^*(\Delta_{k,n}) = \sum_{i \ge 0} (-1)^i \binom{n}{i} \binom{n}{(k-i)d - i}_{k-i},$$
  where the notation $\binom{n}{b}_a$ means the coefficient of t^b in $(1 + t + \cdots + t^{a-1})^n$. Formula contains alternating sums so does not give a combinatorial interpretation itself.

Example

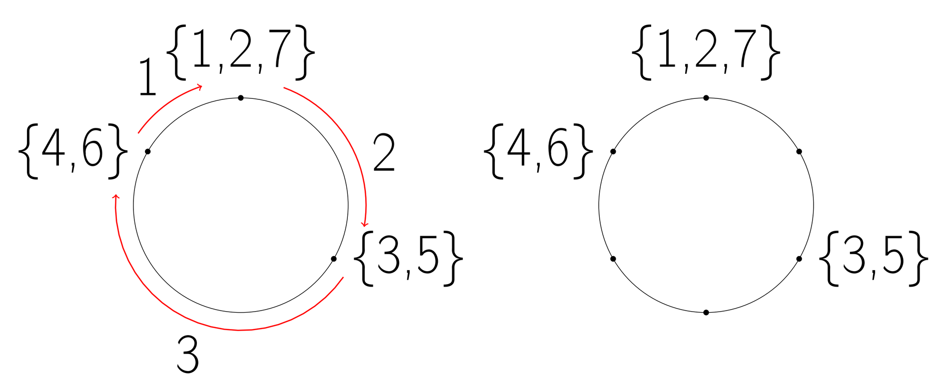

Figure 1. The figure on the left is the picture associated to the decorated ordered set partition $(\{1,2,7\}_2, \{3,5\}_3, \{4,6\}_1)$. This is of type $(6,7)$ and its winding number is 2. It is not hypersimplicial as $3 \ge |\{3,5\}|$. The figure on the right is the picture that contains same information. Instead of denoting the distances, we can insert empty spots to keep distance information.

Definition

A *plate* associated to a decorated ordered set partition $((L_1)_{l_1}, \cdots, (L_m)_{l_m})$ is a characteristic function of the region defined by following inequalities
$$\begin{gathered} x_{L_1} \ge l_1 \\ x_{L_1 \cup L_2} \ge l_1 + l_2 \\ \vdots \\ x_{L_1 \cup \cdots \cup L_m} = l_1 + \cdots + l_m \\ x_i \ge 0, \end{gathered}$$
where $x_S = \sum_{i \in S} x_i$.

- Abve definition is due to Ocneanu who invented Plates theory.

Combinatorial Object and Main Theorem

- Since the numbers $h_d^*(\Delta_{k,n})$ sum up to $A_{k,n-1}$, natural approach is to define a right statistic on permutations so that we can read off the number $h_d^*(\Delta_{k,n})$. However, this approach has not been successful. So we will use a different combinatorial object.

Definition

A *decorated ordered set partition* $((L_1)_{l_1}, \cdots, (L_m)_{l_m})$ of type (k,n) consists of an ordered partition $(L_1, \cdots, L_m)$ of $\{1, 2, ..., n\}$ and an m-tuple $(l_1, \cdots, l_m) \in \mathbb{Z}^m$ such that $l_1 + \cdots + l_m = k$ and $l_i \ge 1$. We call each L_i a *block* and we place them on a circle in a clockwise fashion then think of l_i as the clockwise distance between adjacent blocks L_i and L_{i+1} (indices are considered modulo m). A decorated ordered set partition is called *hypersimplicial* if it satisfies $1 \le l_i \le |L_i| - 1$ for all i and is called *standard* if $1 \in L_1$.

- This object was defined by Ocneanu and standard hypersimplicial decorated ordered set partitions of type (k,n) are known to be in bijection with permutations $w \in S_{n-1}$ with $des(w) = k-1$ (Ocneanu, 2013). However this bijection is rather complicated.
- For a decorated ordered set partition, we define a statistic *winding number*. From the picture associated to a decorated ordered set partition, we start from 1 and go to 2 clockwise then go to 3 clockwise and continue until we come back to 1. And the number of times that total path winds the circle is a winding number.
- Early defined the statistic, winding number, and he conjectured a combinatorial interpretation of $h_d^*(\Delta_{k,n})$ with this ([3]). The conjecture came out from the underlying geometry of a decorated ordered set partition, but the proof is purely combinatorial, using inclusion-exclusion argument.

Theorem

The number of standard hypersimplicial decorated ordered set partitions of type (k,n) with winding number d is $h_d^*(\Delta_{k,n})$.

Results from Plates Theory

- As functions on dialted simplex, plates associated to standard decorated ordered set partitions of type (k,n) are linearly independent (Ocneanu). Span of these functions is denoted as $Pl(k\Delta_{n-1})$.
- Restricting functions on the hypersimplex $\Delta_{k,n}$, standard hypersimplicial decorated ordered set partitions of type (k,n) are linearly independent (Ocneanu). Span of these functions is denoted as $Pl(\Delta_{k,n})$.

- Associated plates for standard decorated ordered set paritions of type (k,n) form a basis for $Pl(k\Delta_{n-1})$.
- Sorting a basis with winding number, we can read off h^*-vector of $k\Delta_{n-1}$.
- Associated plates for standard hypersimplicial decorated ordered set paritions of type (k,n) form a basis for $Pl(\Delta_{k,n})$.
- Sorting a basis with winding number, we can read off h^*-vector of $\Delta_{k,n}$.

References

[1] M. Katzman, The Hilbert series of algebras of Veronese type, Comm. Algebra 33 (2005), 1141-1146.

[2] N. Li, Ehrhart h-vectors of hypersimplices, Discrete Comput. Geom. 48 (2012).

[3] N. Early, Conjectures for Ehrhart h-vectors of hypersimplices and dilated simplices, arXiv:1710.09507.

[4] R. Stanley, Decompositions of rational convex polytopes, Annals of Discrete Math, volume 6 (1980), 333-342.

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