Matroid complexes with prescribed Topology

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Introduction

Given a simplicial complex Δ of dimension $d-1$, let $(f_0, f_1, \dots, f_{d-1})$ be the f-vector, i.e f_i counts the number of i-dimensional faces.
Classification problem: Given a class S of simplicial complexes, what are the possible f-vectors of the elements of S.
Examples:

- Krukal-Katona theorem: f-vectors of general complexes.
- g-theorem: f-vectors of simplicial polytopes (spheres?).
- M-sequences: f-vectors of Cohen-Macaulay complexes.

Simpler: Given that Δ is in S and has dimension $d-1$ bound f_k in terms of f_0 (number of vertices) and understand the extremal cases.
Example: For simplicial polytopes this leads to the stories of stacked and neighborly polytopes, i.e we learn a lot of the geometry of polytopes from answers to the question!

Matroid Complexes

If M is a matroid, then there are three natural complexes associated to them:

1. **Independence complexes:** Independent sets.
2. **Broken circuit complexes:** Associated to an order. Independence complexes containing no broken circuits (removing minimal elements from circuits).
3. **(Order complexes of) Geometric lattices:** Flats/closed sets ordered by inclusion.

The classification problem for matroids is completely out of reach at the moment. Upper an lower bound questions are not too interesting: for example uniform matroids make upper bounds trivial.

Main Theorems

All matroid complexes are shellable thus homotopy equivalent to wedges of equidimensional spheres.
Main Theorem [1]: Fix d, k and a class of complexes (above). The number of complexes in the class homotopy equivalent to a wedge of k spheres of dimension $d-1$ is *finite*.
Can be phrased in many different ways:

- In terms of the top homology group.
- In terms of d and the reduced Euler characteristic, whose absolute value is k.
- In terms of d and $h_d = k$, the top h-number, where

$$\sum_{j=0}^{d} h_j x^d = \sum_{j=0}^{d} f_{j-1} x^j (1-x) d - j$$

Idea: It may be a good idea to change the parameters for matroid complexes in the upper/lower bound questions.

Independence Complexes

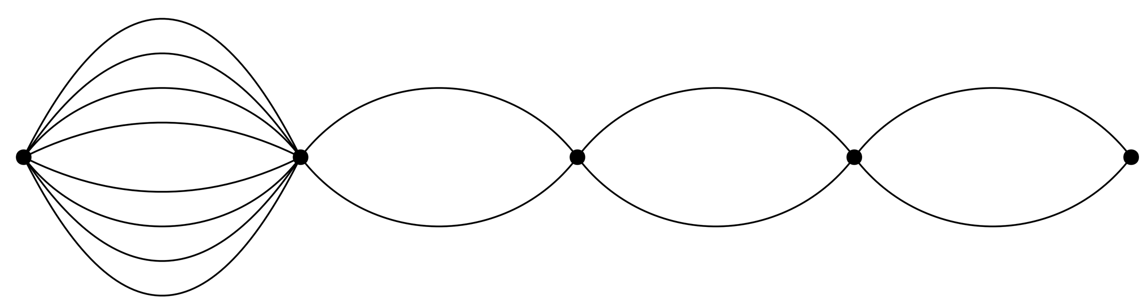
There are several ways to understand the phenomenon for independence complexes:

1. **Homology bases:** Björner constructed a basis for the homology that covers the complex and consists of spheres with finitely many combinatorial types.
2. **Internal activity poset:** The internal activity poset ordered by inclusion has exactly k maximal elements (Dawson 84, Las Vergnas). Bounds the number of vertices!
3. **h-vector decompostions:** Explicit bounds for the number of such matroids in terms matroids with k bases.
4. **Convex ear decomposition:** Chari (97) showed that the independence complex can be constructed by attaching very special kinds of handles (PS-balls) to an initial sphere (PS-sphere) as we explain below.

PS balls and spheres: Joins of boundaries of simplices and one (possibly empty) simplex. A **PS Ear decompositions** of a $d-1$-dimensional simplicial complex is a collection of complexes $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_{k-1}$, where Γ_0 is a PS-Ball and Γ_{i+1} is obtained from Γ_i by attaching a PS-ball along the boundary.
Theorem [2]: Independence complexes admit PS-ear decompositions.
Theorem: From the PS-ear decomposition one gets the finiteness theorem and a characterization of the extremal upper bound matroids! In particular:

$$h_i(\mathcal{I}(M)) \le \binom{d}{i} + (k-1)\binom{d-1}{i-1}, \qquad 0 \le i \le d. \tag{1}$$

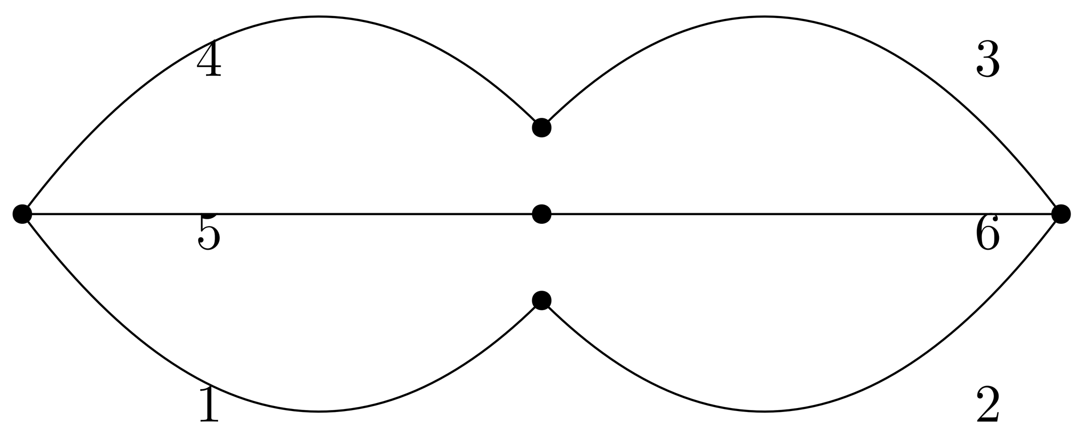
The following is the *unique* maximizer when $d=4$ and $k=7$.



Note: The unique elements that maximize all f and h entries simultaneously, are not simple. Restricting to simple matroids is a challenging task!

Broken Circuit Complexes

Example Consider the graph



The circuits are [1234], [1256], [3456] so the broken circuits are [234], [256], [456]. The bases containing **no broken circuits** are

[1245], [1246], [1235], [1236], [1345], [1346], [1356].

The h-vector is $(1, 2, 3, 1, 0)$.
All independence complexes are broken circuit complexes, but the converse fails.
Main issue: Most of the techniques break down! No nice homology bases, or well behaved internal activity, or Convex-ear decompositions! Nevertheless
Theorem [3] The number of broken circuit complexes homotopy equivalent to a wedge of k spheres of dimension $d-1$ is finite.

References

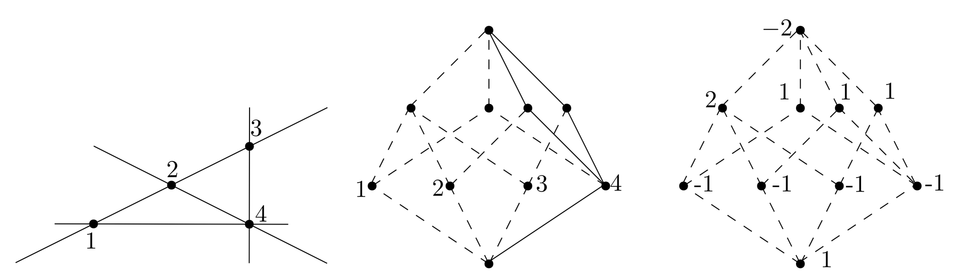
[1] F. Castillo and J. Samper. *Finiteness theorems for matroid complexes with prescribed topology*, 2018.

[2] M. K. Chari. Two decompositions in topological combinatorics with applications to matroid complexes. *Trans. Amer. Math. Soc.*, 349(10):3925–3943, 1997.

[3] E. Swartz. Lower bounds for h-vectors of k-CM, independence, and broken circuit complexes. *SIAM J. Discrete Math.*, 18(3):647–661, 2004/05.

Geometric Complexes

The following is an example of an affine matroid with the corresponding lattice of flats.



In this case there is a stronger statement than in the Main Theorem:
Theorem Fix a natural number k. There exist finitely many geometric lattices $L_1, \cdots, L_m$ such that if L is any finite geometric lattice satisfying $|\tilde{\chi}(\mathcal{O}(L))| = k$ then $L = L_i \times B_d$ for some i, d.
Note: For geometric lattices the top $h-$ number is the mobius function on the whole poset.
This can be computed using EL-labelings.

Further Questions

- For independece complexes what are the upper bounds for *simple* matroids?
- What are the equality cases for the upper bounds in Broken Circuit complexes?
- Is there an algorithm that generates all matroids of a given rank and topology efficiently for some (hopefully not very small) parameters?