Deformations of Coxeter permutahedra and Coxeter submodular functions

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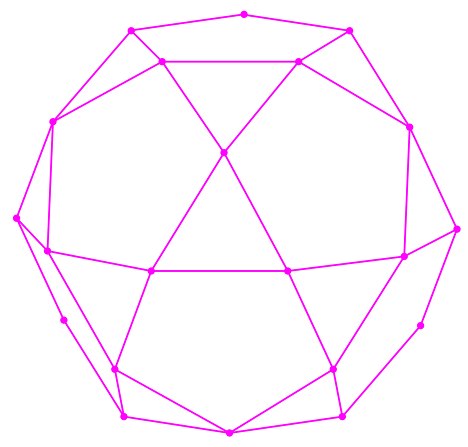


Deformations

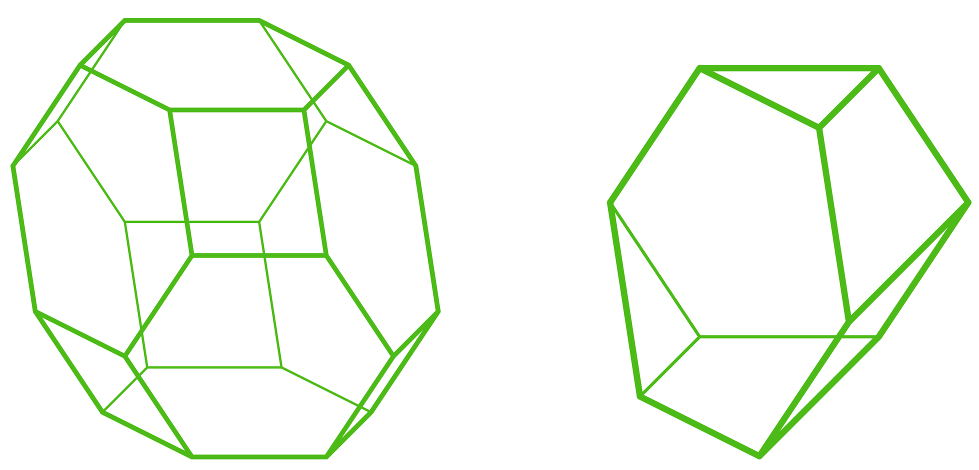
A polyhedron Q is a *deformation of* P if the normal fan Σ_Q is a coarsening of the normal fan Σ_P.

When P is a simple polytope, it is shown in [4, Theorem 15.3] that we may think of the deformations of P equivalently as being obtained by any of the following three procedures:

• moving the vertices of P while preserving the direction of each edge, or

• changing the edge lengths of P while preserving the direction of each edge, or

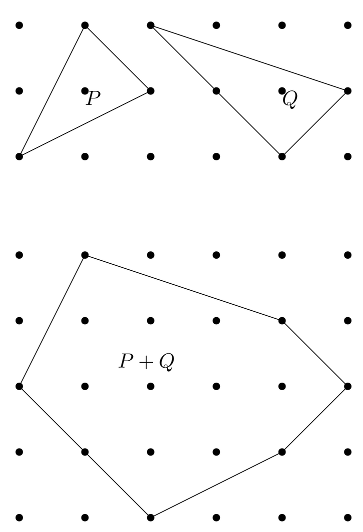
• moving the facets of P while preserving their directions, without allowing a facet to move past a vertex.



The standard 3-permutahedron and one of its deformations.

The *Minkowski sum* of two polytopes Q and R in the same vector space V is the polytope

$$P + Q := \{p + q \,:\, p \in P, q \in Q\}.$$



P is a deformation of $P+Q$. This is, up to scaling, the only source of deformations. For this reason, deformations of polytopes are also often called *weak Minkowski summands*.

Theorem (Shepard [2]): If P and Q be polytopes, then Q is a deformation of P if and only if there exist a polytope R and a scalar $\lambda > 0$ such that $Q + R = \lambda P$.

Zonotopes

Let $\mathcal{A} = \{v_1, \ldots, v_m\} \subset V$ be a set of vectors and let $\mathcal{H} = \{H_1, \ldots, H_m\}$ be the corresponding hyperplane arrangement in U given by the hyperplanes $H_i = \{u \in U \,:\, \langle u, v_i\rangle = 0\}$ for $1 \le i \le m$. The hyperplane arrangement $\mathcal{H}$ then determines a fan $\Sigma_{\mathcal{H}}$ whose maximal cones are the closures of the connected components of the arrangement complement.

Let $\mathcal{A} = \{v_1, \ldots, v_m\} \subset V$. The *zonotope* of $\mathring{A}$ is the Minkowski sum

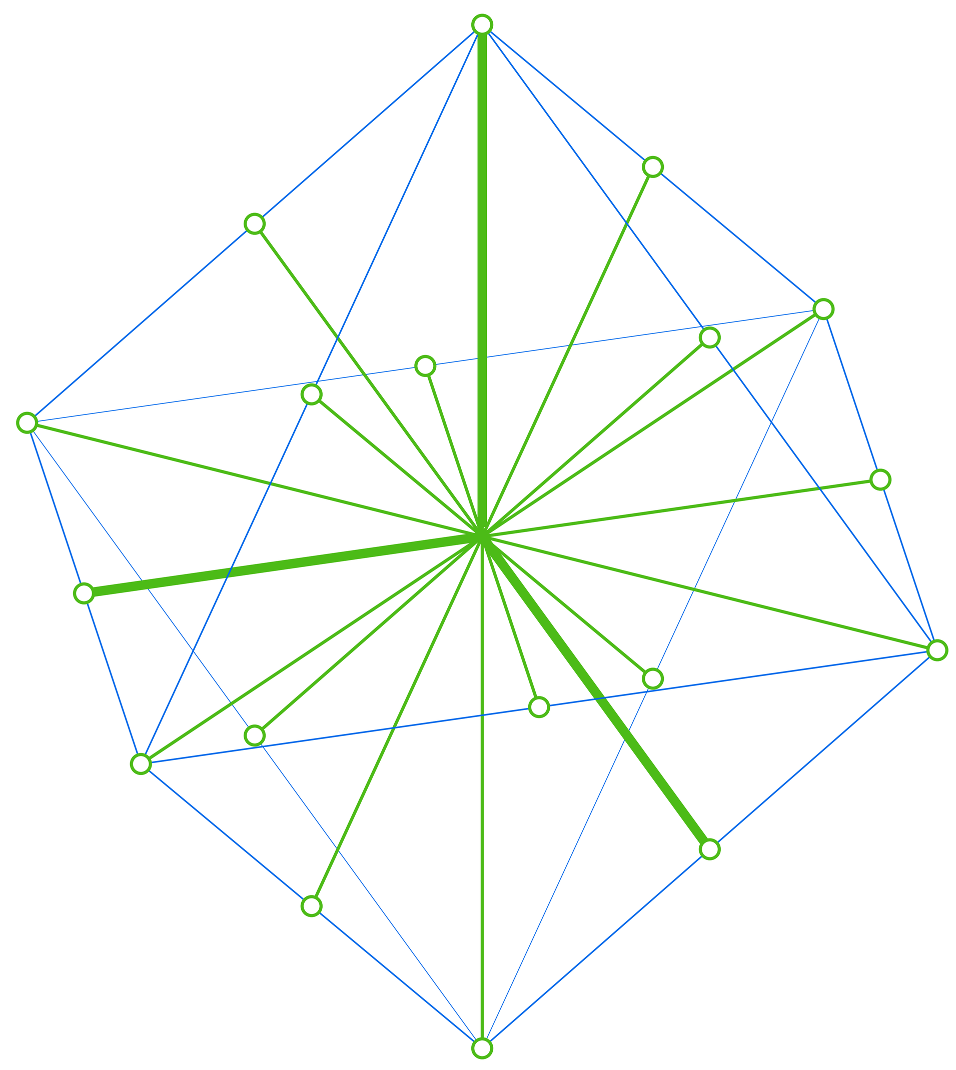
$$\mathcal{Z}(\mathcal{A}) := [0, v_1] + \cdots + [0, v_m].$$

We can describe the (extended) deformations of $\mathcal{Z}(\mathcal{A})$ easily as follows.

Proposition Let $\mathcal{A}$ be a finite set of vectors in V. A polytope is a deformation of the zonotope $\mathcal{Z}(\mathcal{A})$ if and only if every edge is parallel to some vector in $\mathcal{A}$.

Root Systems

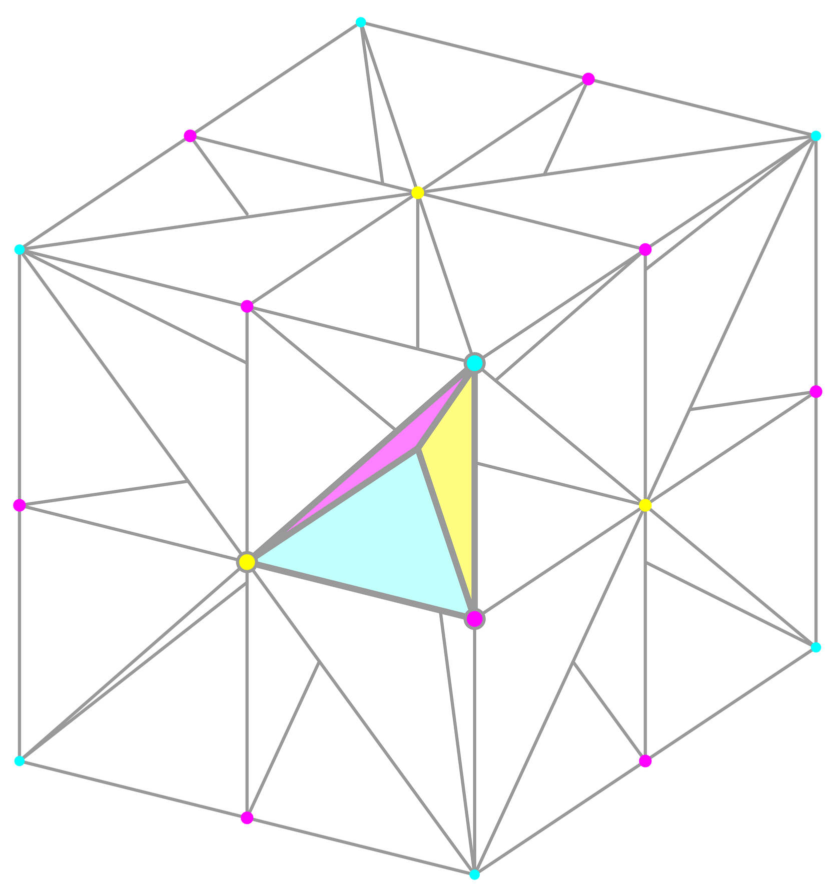
Root Systems are highly symmetric vector configurations.



The root system B_3

Coxeter Complex

Given a root system, the Coxeter complex is the fan obtained with slicing the whole space with the hyperplanes perpendicular to the roots.



The group generated by all the reflections through hyperplanes is *finite* and denoted W (for Weyl group).

We can arbitrarily choose one region and call it the *Fundamental Chamber*. The rays spanning that region are called the fundamental weights. Any element obtained by applying W to the fundamental weight is call a weight.

Coxeter submodular functions

We can parametrize the space of all deformations of the Coxeter complex. We do this using functions on the (finitely many!) weights as parameters.

Theorem [1]: Let Φ be a finite root system with Weyl group W and $\mathcal{R} = W\{\lambda_1, \ldots, \lambda_d\}$ be the set of W-conjugates of fundamental weights $\lambda_1, \ldots, \lambda_d$. The deformations of the Φ-permutahedron are in bijection with the Φ-*submodular functions* $h :\to$ that satisfy the following inequalities:

For every element $w \in W$, every simple reflection s_i, and corresponding fundamental weight λ_i,

$$h(w\lambda_i) + h(ws_i\lambda_i) \ge \sum_{j \in N(i)} -A_{ji}\, h(w\lambda_j) \qquad (1)$$

where $N(i)$ is the set of neighbors of i in the Dynkin diagram and A is the Cartan matrix.

Furthermore, all such inequalities are *facet defining* and we can count them.

Type A

In Type A the theorem recovers the known result that generalized permutohedra are in bijection with *submodular funtions* i.e., with functions $f : 2^{[n]} \to \mathbb{R}$ such that

$$f(A) + f(B) \ge f(A \cap B) + f(A \cup B)$$

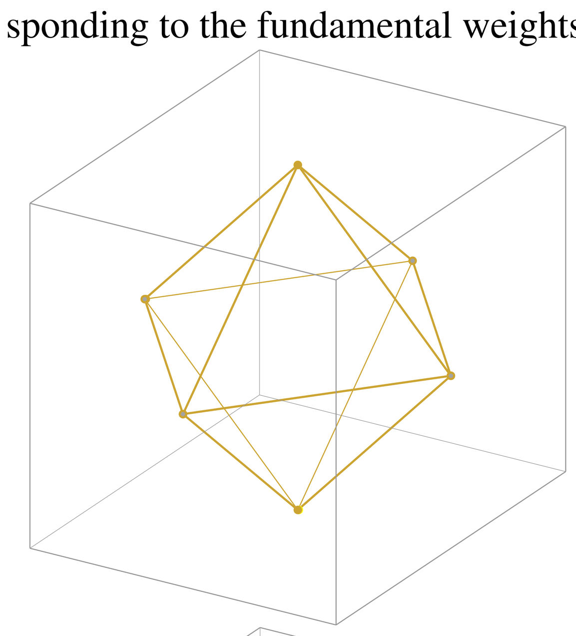
There are many examples of generalized permutohedra. One of the most studied are matroid polytopes

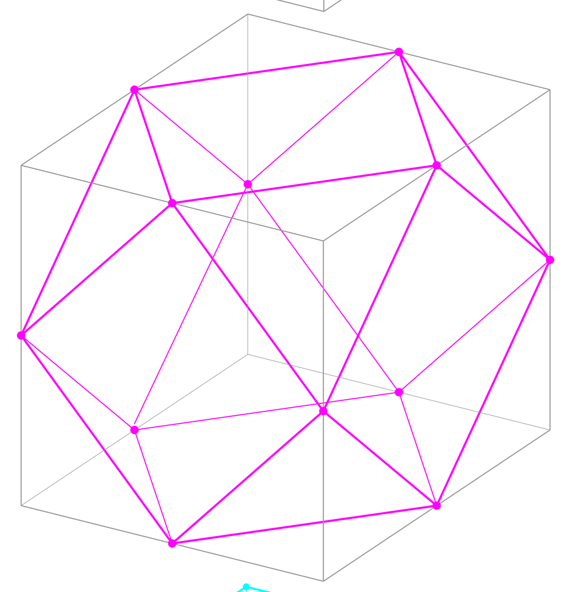
Theorem (Nguyen [3]) Connected matroid polytopes are extremal rays of the submodular cone. In other words, they cannot be decomposed as minkowski sum of two other polytopes (unless one of them is a point).

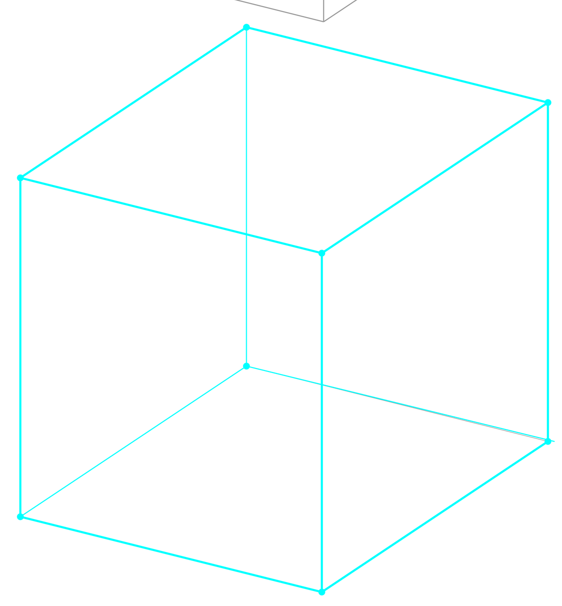
We can describe some extremal rays.

Fundamental weight polytopes

The *fundamental weight polytopes* or Φ-*hypersimplices* of the root system Φ are the d weight polytopes $P_\Phi(\lambda_1), \ldots, P_\Phi(\lambda_d)$ corresponding to the fundamental weights of Φ.







Theorem [1]:A weight polytope P of a crystallographic root system Φ is indecomposable if and only if $P = kP_\Phi(\lambda_i)$ for $k > 0$ and a fundamental weight λ_i such that the edges adjacent to i in the Dynkin diagram are unlabeled; that is, the Dynkin diagram $\Gamma(\Phi_{N(i)\cup i})$ is simply laced.

Further Questions

In type A, every generalized permutahedron in ^{d} is a signed Minkowski sum of the simplices $\Delta_S = \mathrm{conv}(e_s \,:\, s \in S)$ for $S \subseteq [d]$. Geometrically, this corresponds to the statement that the $2^d - 1$ polytopes Δ_S, which are rays of the (2^d-1)-dimensional submodular cone, are also a basis for 2^{d-1}. Remarkably, one may compute the mixed volumes of these polytopes P_S, and this gives combinatorial formulas for the volume of any generalized permutahedron.

Is there a similarly nice choice of rays of the Φ-submodular cone that generate all others? Can one compute their mixed volumes?

References

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