Cyclic Sieving, Necklaces, and Bracelets

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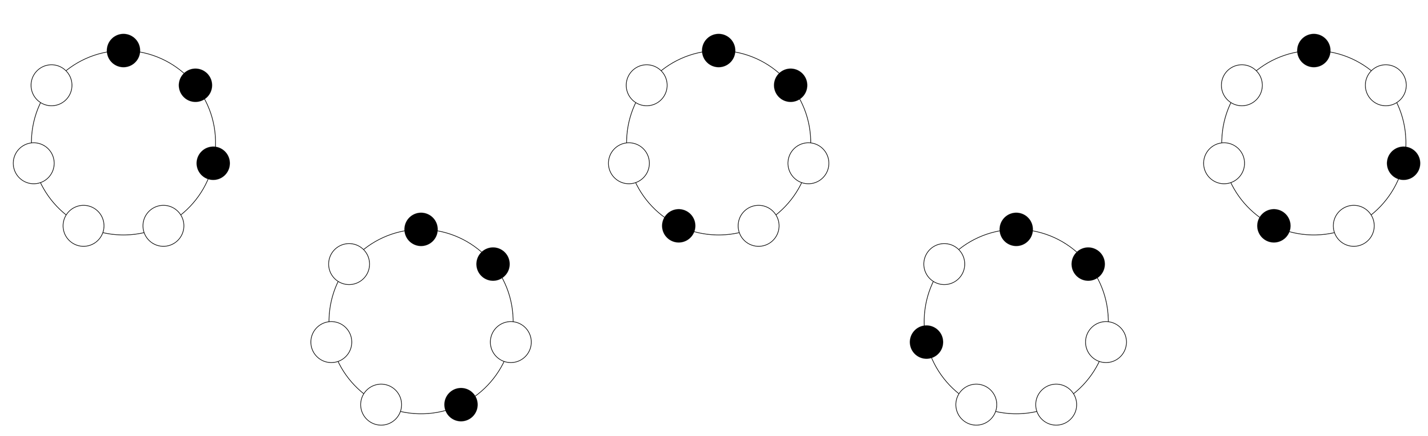
Background

Given a sequence $\alpha = (\alpha_1, \dots, \alpha_r)$ of nonnegative integers that sums to n, the **multinomial coefficient**

$$\binom{n}{\alpha} = \binom{n}{\alpha_1, \dots, \alpha_r} = \frac{n!}{\alpha_1! \cdots \alpha_r!}$$

is a positive integer, counting the set X_α of words having exactly α_i occurrences of the letter i for each $i = 1, 2, \dots, r$. The symmetric group S_n acts on the set of such words by permuting positions, and when restricting this action to the cyclic subgroup $C = \langle c \rangle$ generated by $c = (1, 2, \dots, n)$, the orbits are called α-**necklaces**. The C-action on X_α will be free if and only if $\gcd(\alpha) = \gcd(\alpha_1, \dots, \alpha_r) = 1$, and thus the number of α-necklaces in this case is given by $C(\alpha) = \frac{1}{n}\binom{n}{\alpha}$.

When $\alpha = (a, a+1)$, this is the well-known Catalan number. For example, when $\alpha = (3, 4)$, there are $C(3,4) = \frac{1}{7}\binom{7}{3} = \frac{1}{4}\binom{6}{3} = 5$ such necklaces with 3 black beads and 4 white beads, shown here:



We will write $C(\alpha; q)$ to mean the natural q-analogue of $C(\alpha)$; that is:

$$C(\alpha; q) = \frac{1}{[n]_q} \begin{bmatrix} n \\ \alpha \end{bmatrix}_q .$$

In their paper defining cyclic sieving, Reiner, Stanton, and White showed that $C(\alpha; q)$ is a polynomial whenever $\gcd(\alpha) = 1$.

Cyclic Sieving

Recall that for a cyclic group $C = \langle \tau \rangle$ of order m acting on a set X, and a polynomial $f \in \mathbb{Z}[q]$ (not necessarily nonnegative), the triple (X, f, C) **exhibits the cyclic sieving phenomenon** if for every integer d we have that $|\{x \in X : \tau^d(x) = x\}| = f(e^{\frac{2\pi i}{m}})$.

Theorem. Fix a positive integer $m \geq 2$, and suppose that either $n \equiv 1 \bmod m$, or n is even and $n \equiv 2 \bmod m$. Let α be a sequence of nonnegative integers with $\gcd(\alpha) = 1$, which is not $(\ell, \ell, \dots, \ell, 2)$ for any divisor ℓ of m. Further assume that $\tau \in N_{S_n}(C)$ has order m and cycle type

$$\mathrm{cyc}(\tau) = \begin{cases} (m^{\frac{n-1}{m}}, 1) & \text{if } n \equiv 1 \bmod m, \\ (m^{\frac{n-2}{m}}, 1, 1) & \text{if } n \equiv 2 \bmod m. \end{cases}$$

Then the triple $(C \backslash X_\alpha, C(\alpha; q), \langle \tau \rangle)$ exhibits the cyclic sieving phenomenon, where $\langle \tau \rangle \cong \mathbb{Z}/m\mathbb{Z}$ acts on $C \backslash X_\alpha$ via $\tau C w = C(\tau w)$.

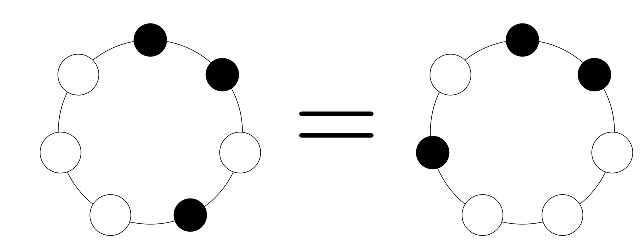
In particular, there are involutions $\tau \in N_{S_n}(C)$ which act on necklaces by reflection; orbits for this τ-action are called **bracelets**. We say that a bracelet is **asymmetric** if it is a τ-orbit of necklaces of size two.

Corollary. When $\gcd(\alpha) = 1$, the set of α-necklaces, the polynomial $C(\alpha; q) = \sum_i a_i q^i$, and the τ-action by reflection exhibits the cyclic sieving phenomenon. That is, the coefficient sums $a_0 + a_2 + a_4 + \cdots$ and $a_1 + a_3 + a_5 + \cdots$ count the total number of bracelets, and the number of asymmetric bracelets, respectively.

In the example of $\alpha = (3, 4)$, we have

$$C(\alpha; q) = \frac{1}{[7]_q} \begin{bmatrix} 7 \\ 3 \end{bmatrix}_q = 1 + q^2 + q^3 + q^4 + q^6,$$

with $a_0 + a_2 + a_4 + a_6 = 4$ and $a_1 + a_3 + a_5 = 1$. This agrees with the fact that the five necklaces shown above give rise to four bracelets, only one of which is asymmetric, namely the bracelet shown here:



Parity-Unimodality

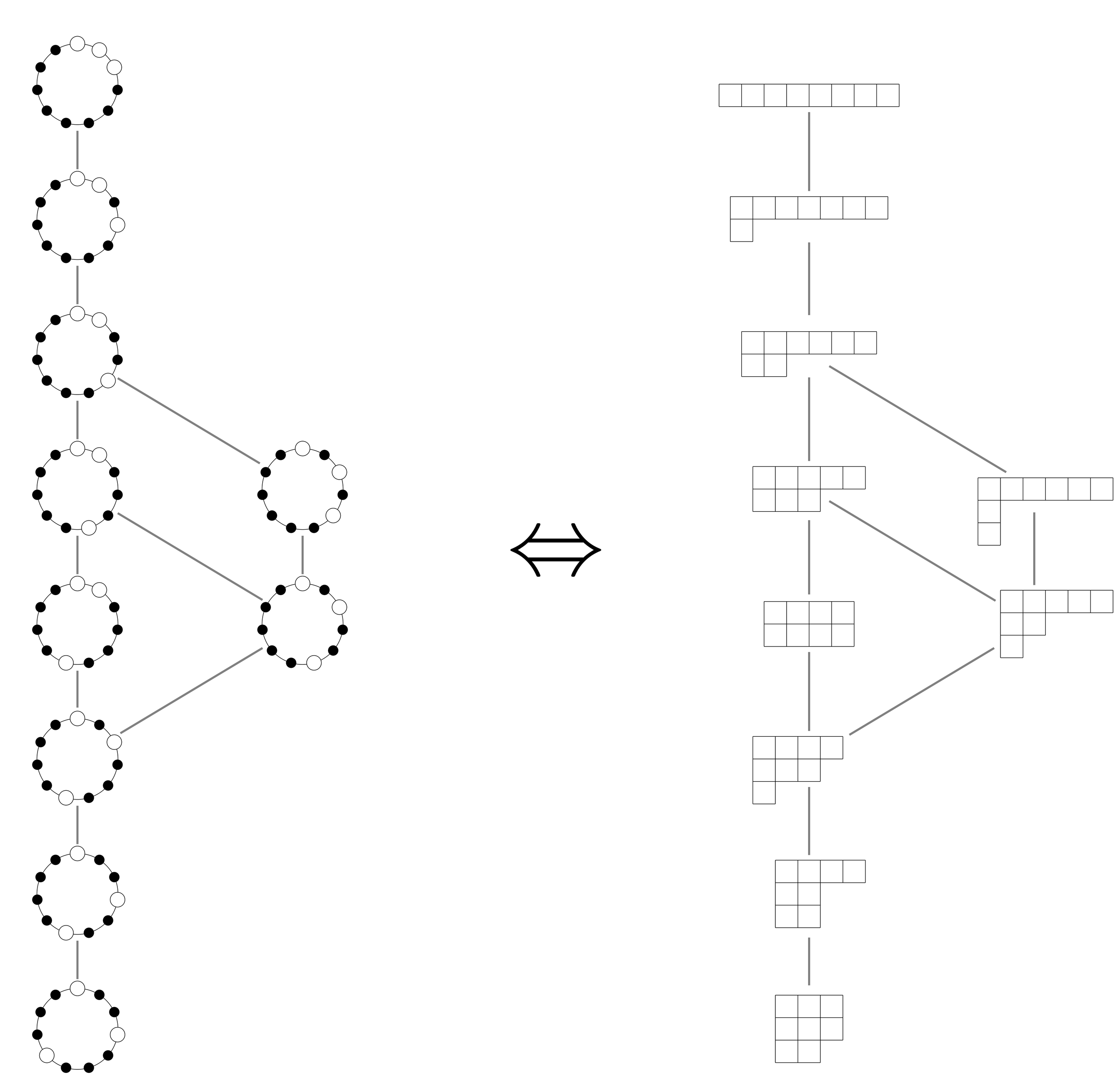
Say that a polynomial $X(q) = \sum_i a_i q^i$ is **parity-unimodal** if both subsequences $(a_0, a_2, a_4, \dots)$ and $(a_1, a_3, a_5, \dots)$ are unimodal.

Conjecture. $C(\alpha; q)$ is parity-unimodal when $\gcd(\alpha) = 1$.

This conjecture has been checked for all relevant compositions α of $n \leq 30$. Moreover, known results in the theory of rational Cherednik algebras imply the conjecture for certain three-term sequences:

Theorem. Let a, b, and k be positive integers satisfying $\gcd(a, b) = 1$ and $0 \leq k \leq a < b$. Then the "rational q-Schröder polynomial" $C(k, a-k, b-k; q)$ is parity-unimodal.

Together with a result of Proctor, this suggests that there may be two "natural" Peck posets, on rational Schröder bracelets and asymmetric bracelets, whose rank sizes are $(a_0, a_2, a_4, \dots)$ and $(a_1, a_3, a_5, \dots)$.



When $k = 0$ and $a = 3$, these bracelets are in natural bijection with partitions λ of b with at most 3 rows. These partitions form a ranked interval in the dominance order with has rank sizes $(a_0, a_2, a_4, \dots)$. Moreover, restricting to the odd bracelets also gives a ranked poset, with rank sizes $(a_1, a_3, a_5, \dots)$, and both of these posets have symmetric chain decompositions. The interval for $b = 8$ is shown above.

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