Multiplicities of Schubert varieties in the symplectic flag varieties

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Objectives

The **multiplicity** of a point on a Schubert variety is a positive integer which depends on two elements of the Weyl group. **What is a combinatorial formula for this number?** We give an answer for vexillary Schubert varieties in the symplectic flag variety.

Previous research

- Schubert varieties in Grassmannian [4]
- Vexillary Schubert varieties in flag variety [5]
- Schubert varieties in Lagrangian Grassmannian [2][3]

Basic Notation

- $W_n \cong S_n \rtimes \{\pm 1\}^n$, the group of *signed permutations*.
- $V = \mathbb{C}^{2n}$ with a non-degenerate skew symmetric bilinear form $\langle \cdot, \cdot \rangle$.
- $Fl_n^C = \{E_\bullet \mid E_\bullet : E_n \subset \cdots \subset E_1 \subset V \text{of isotropic subspaces, } \dim E_i = n+1-i\}$.

Vexillary Signed Permutations and Schubert Varieties

A *triple* is $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ where $\mathbf{k} : 0 < k_1 < \cdots < k_s$, $\mathbf{p} : p_1 \geq \cdots \geq p_s > 0$, and $\mathbf{q} : q_1 \geq \cdots \geq q_s > 0$ satisfy $(p_i - p_{i+1}) + (q_i - q_{i+1}) > k_{i+1} - k_i$ for all $i = 1, \ldots, s-1$.

We can construct $w \in W_n$ from a triple τ in a certain way.

A signed permutation $w \in W_n$ is *vexillary* if it can be constructed as $w = w(\tau)$, for some triple τ.

The *Schubert variety* Ω_w is defined by rank conditions for a general permutation $w \in W_n$. It has codimension $\ell(w)$ in the symplectic flag variety.

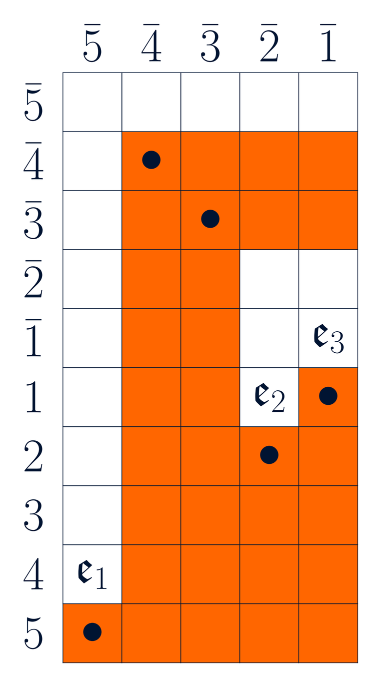
The *Schubert variety* associated to a vexillary permutation w is defined by

$$\Omega_w = \{E_\bullet \mid \dim(E_{p_i} \cap F_{q_i}) \geq k_i \text{ for } i = 1, \ldots, s\} \subseteq Fl_n^C,$$

where $F_\bullet$ is a fixed isotropic flag.

Example

$w = \bar{1}\,\bar{2}\,3\,4\,\bar{5}$



The *essential set* $\mathcal{E}ss(w)$ corresponding to a vexillary signed permutation w consists of the southeast corners of the diagram; it gives a minimal list of rank conditions [1]. In the above example, it is the boxes labelled $\mathfrak{e}_1, \mathfrak{e}_2$ and $\mathfrak{e}_3$

Hilbert-Samuel Multiplicity

Let X be an algebraic variety containing a point p. Let $R = \mathcal{O}_{X,p}$ be the local ring of X at p with the maximal ideal $\mathfrak{m}$.

The *Hilbert-Samuel polynomial* of R is for $n \gg 0$

$$\mathcal{P}_R(n) = \dim_{\mathbb{C}}(R/\mathfrak{m}^n) = (m/d!)\, x^d + \cdots,$$

where $d = \dim R$, $m \in \mathbb{Z}_{>0}$.

The *Hilbert-Samuel multiplicity* of R is

$$\mathrm{mult}_p(X) = m$$

Inner and Outer Shapes

We define strict partitions by vexillary permutation w and signed permutation v such that $v \geq w$. Let $r_w(\mathfrak{e})$ count the dots in the north-west corner of $\mathfrak{e}$ of the diagram associated to w.

Shape of a vexillary permutation $\lambda(w)$

- Obtain the box $\mathfrak{e}'$ by moving diagonally north-west by $r_w(\mathfrak{e})$ units for each $\mathfrak{e} \in \mathcal{E}ss(w)$
- Denote λ by the smallest shifted diagram containing all the $\mathfrak{e}'$ with $\mathfrak{e} \in \mathcal{E}ss(w)$.

Outer shape μ

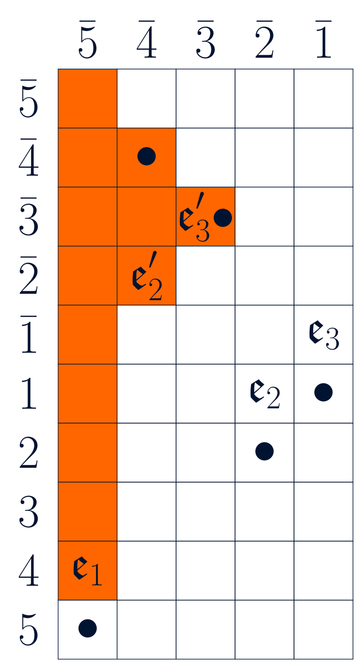
- Obtain the box $\mathfrak{e}'$ by moving diagonally north-west by $r_v(\mathfrak{e})$ units for each $\mathfrak{e} \in \mathcal{E}ss(w)$
- Denote μ by the smallest shifted diagram containing all the $\mathfrak{e}'$ with $\mathfrak{e} \in \mathcal{E}ss(w)$.

Example

We describe $\lambda(w)$ and μ from w and v as follows:

Shape of a vexillary permutation

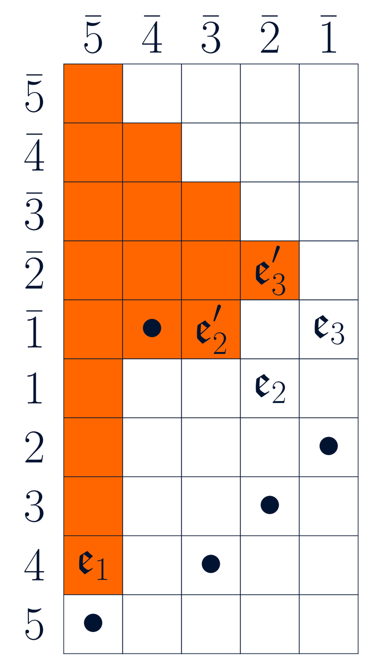
$w = \bar{1}\,\bar{2}\,3\,4\,\bar{5}$



$\lambda = (9, 3, 1)$.

Outer shape

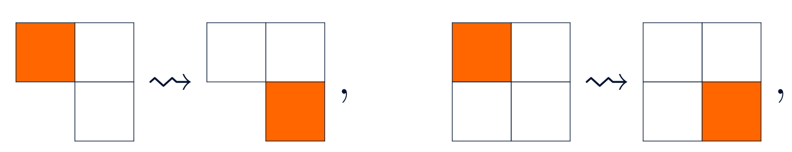
$w = \bar{1}\,\bar{2}\,3\,4\,\bar{5}$, $v = \bar{2}\,\bar{3}\,\bar{4}\,1\,\bar{5}$



$\mu = (9, 4, 3, 1)$.

Excited Young Diagram

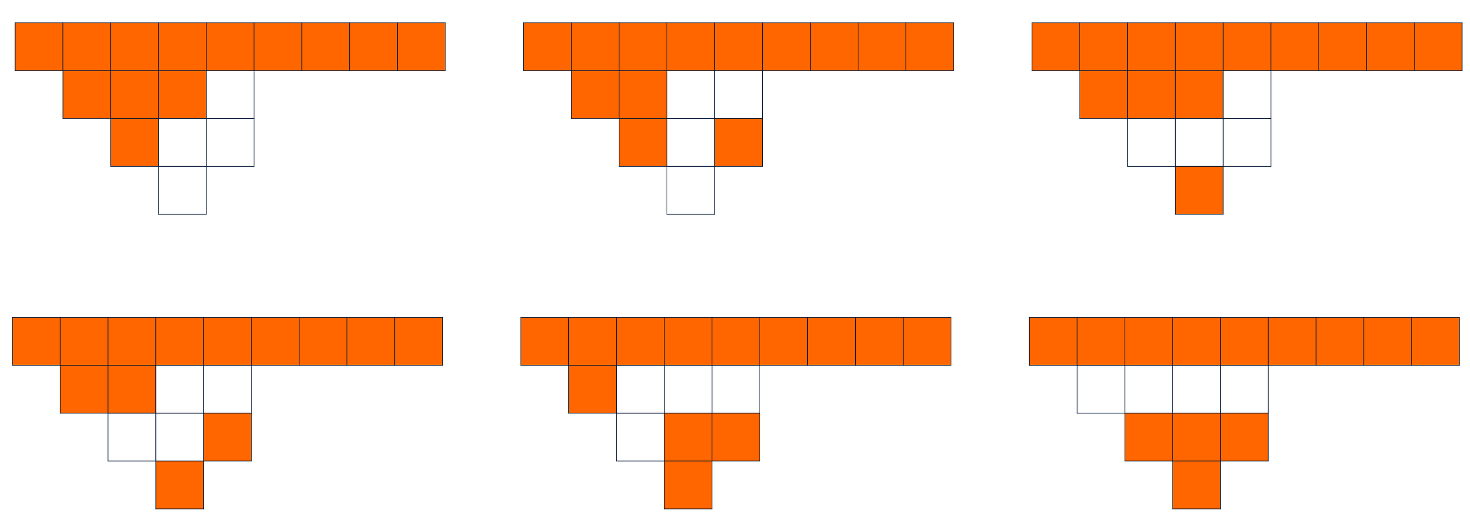
Given two strict partitions μ and λ, let $\mathcal{E}_\mu(\lambda)$ be the set of diagrams that is obtained by the following successive applications of elementary excitations.



By successive applications of elementary excitations, we obtain the multiplicity of the Schubert variety Ω_w in Lagrangian Grassmannian at the point e_v. [3]

Example

$w = \bar{1}\,\bar{2}\,3\,4\,\bar{5}$, $v = \bar{2}\,\bar{3}\,\bar{4}\,1\,\bar{5}$.



$\Rightarrow \mathrm{mult}_{e_v}(\Omega_w) = 6$

Theorem on Multiplicity

Let w be vexillary and $v \in W_n$ such that $w \leq v$. For $\lambda = \lambda(w)$ and μ from the pair (v, w), the Hilbert-Samuel multiplicity is given by

$$\mathrm{mult}_{e_v}(\Omega_w) = \#\mathcal{E}_\mu(\lambda).$$

Our proof reduces to the (Lagrangian) Grassmannian case, and also gives a new and simple argument for the type A case considered by Li-Yong.

Future work

Extend the result to the other type B and D.

References

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[5] L. Li and A. Yong, *Some degenerations of Kazhdan-Lusztig ideals and multiplicities of Schubert varieties*, Adv. Math. **229**, (2012), no. 1, 633-667.