

**Birational Antichain Toggling and Rowmotion** Michael Joseph\* and Tom Roby

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**Products of Two Chains Posets** 

running examples.

The posets  $[2] \times [2]$  and  $[2] \times [3]$  serve as



# **Summary of Main Ideas**

- Rowmotion and its various liftings are maps of longstanding interest in dynamical algebraic combinatorics, which have been studied for their periodicity, homomesy, cyclic sieving, and resonance.
- ► We lift antichain rowmotion to the piecewise-linear and birational realms, in parallel with the construction for order ideals.
- We construct an explicit bijection between the two (antichain and order) birational toggle groups, lifting a map from the combinatorial realm. This yields an equivariant bijection between BAR-motion and BOR-motion. In particular, BAR-motion has the same order as BOR-motion.
- Antichain rowmotion on the poset  $[a] \times [b]$  rotates the Stanley–Thomas word, proving fiber homomesy [PR15]. We define a birational lifting of the Stanley–Thomas word to prove a fiber homomesy holds at the birational level as well.
- Equality of expressions proven in the birational realm imply the corresponding results in the piecewise-linear and combinatorial realms.

## **Rowmotion in the Combinatorial Realm**

**Combinatorial rowmotion** is a particular permutation of the set of order ideals  $\mathcal{J}(P)$  of a finite poset *P* or of the set of antichains  $\mathcal{A}(P)$  of *P*. It is studied for its remarkable properties (periodicity, homomesy, cyclic sieving, resonance) on certain families of posets (especially products of chains and root posets). It was first studied as a map on  $\mathcal{A}(P)$  by Brouwer and Schrijver [BS74], and goes by several names; in recent literature, the name "rowmotion," due to Striker and Williams [SW12] (who summarize the history), has stuck.

#### **Definitions: Rowmotion and Transfer Maps**

We follow the notation of Einstein and Propp [EP18] to define natural bijections between the sets  $\mathcal{J}(P)$  of all *order ideals* of P,  $\mathcal{F}(P)$  of all *order filters* of P, and  $\mathcal{A}(P)$  of all *antichains* of P.

# **Explicit Isomorphism Between Antichain and Order Birational Toggle Groups**

We have an isomorphism from the **birational antichain toggle group**  $\operatorname{BTog}_A(P)$  generated by  $\{\tau_e : e \in P\}$  and the **birational order toggle group**  $\operatorname{BTog}_O(P)$  generated by  $\{T_e : e \in P\}$ . The isomorphism is given by  $\tau_e \mapsto \tau_e^*$  with inverse given by  $T_e \mapsto T_e^*$ .

For  $e \in P$ , we define the following.

- ► Let  $T_e^* \in \operatorname{BTog}_A(P)$  by  $T_e^* := \tau_{e_1} \tau_{e_2} \cdots \tau_{e_k} \tau_e \tau_{e_1} \tau_{e_2} \cdots \tau_{e_k}$ , where  $e_1, \ldots, e_k$  are the elements of *P* covered by *e*. (If *e* is a minimal element of *P*, then  $t_e^* = \tau_e$ .)
- Let  $\eta_e \in \operatorname{BTog}_O(P)$  by  $\eta_e := T_{x_1}T_{x_2}\cdots T_{x_k}$  where  $(x_1, x_2, \dots, x_k)$  is a linear extension of the subposet  $\{x \in P : x < e\}$  of *P*.
- Let  $\tau_e^* := \eta_e T_e \eta_e^{-1} \in \operatorname{BTog}_O(P)$ .

#### Theorem (J.–R.)

Let  $e \in P$ . The diagrams below commute on the domains in which the maps are defined.





- The map  $\Theta: 2^P \to 2^P$  where  $\Theta(S) = P \setminus S$  is the **complement** of *S* (sending order ideals to order filters and vice versa).
- ► The up-transfer  $\Delta : \mathcal{J}(P) \to \mathcal{A}(P)$ , where  $\Delta(I)$  is the set of maximal elements of *I*. For an antichain  $A \in \mathcal{A}(P)$ ,  $\Delta^{-1}(A) = \{x \in P : x \leq y \text{ for some } y \in A\}.$
- ► The **down-transfer**  $\nabla$  :  $\mathcal{F}(P) \to \mathcal{A}(P)$ , where  $\nabla(F)$  is the set of minimal elements of *F*. For an antichain  $A \in \mathcal{A}(P)$ ,  $\nabla^{-1}(A) = \{x \in P : x \ge y \text{ for some } y \in A\}.$

**Order ideal rowmotion** is the map  $\rho_{\mathcal{J}} : \mathcal{J}(P) \to \mathcal{J}(P)$  given by the composition  $\rho_{\mathcal{J}} = \Delta^{-1} \circ \nabla \circ \Theta$ . **Antichain rowmotion** is the map  $\rho_{\mathcal{A}} : \mathcal{A}(P) \to \mathcal{A}(P)$  given by the composition  $\rho_{\mathcal{A}} = \nabla \circ \Theta \circ \Delta^{-1}$ .

## **Example of Rowmotion as 3-Step Processes** ( $P = [2] \times [2]$ )

In each step, the elements of the subset of the poset are given by the filled-in circles.

$$\rho_{\mathcal{J}}: \bigoplus_{\bullet} \xrightarrow{\Theta} \xrightarrow{\Theta} \bigoplus_{\leftarrow} \bigvee_{\bullet} \bigoplus_{\leftarrow} \bigoplus_{\leftarrow} \xrightarrow{\Delta^{-1}} \bigoplus_{\bullet} \bigoplus_{\bullet} \bigoplus_{\leftarrow} \bigoplus_{\oplus$$

## **Generalizing Antichain Rowmotion to Chain Polytopes and Birational Labelings**

For a poset *P*, Stanley's chain polytope C(P) is the set of [0, 1]-node-labelings such that the sum of the labels along any chain is at most 1 [Sta86]. We extend rowmotion on  $\mathcal{A}(P)$  (the vertices of C(P)) to all of C(P) [EP18, Jos19].

## **The Poset** $\widehat{P}$

So that multisets in the definitions (right) are nonempty, we extend the poset *P* to  $\widehat{P}$  by adjoining a minimal element  $\widehat{0}$  and maximal element  $\widehat{1}$ .

#### **Piecewise-Linear to Birational**

We lift to the birational realm, by "detropicalizing" the operations through the following replacements. Note that *C* is just a generic constant

## **Chain Polytope Rowmotion [EP18, Jos19]**

We define  $\rho_{\mathcal{C}} : \mathcal{C}(P) \to \mathcal{C}(P)$  as the composition  $\rho_{\mathcal{C}} = \nabla \circ \Theta \circ \Delta^{-1}$  where •  $(\Theta f)(x) = 1 - f(x),$ •  $(\nabla f)(x) = f(x) - \max_{y \leq x} f(y) \left( \text{with } f\left(\widehat{0}\right) = 0 \right),$ 

•  $(\Delta^{-1}f)(x) = \max \{f(y_1) + f(y_2) + \dots + f(y_k) : x = y_1 \leqslant y_2 \leqslant \dots \leqslant y_k \leqslant \hat{1}\}.$ The notations  $y \ge x$  and  $y \leqslant x$  indicate *covering relations* in *P*.

#### **Birational Antichain Rowmotion [EP18]**

Let  $\mathbb{K}^P$  denote the set of node-labelings of *P* with elements of a field  $\mathbb{K}$ . We define **birational antichain rowmotion (BAR-motion)** as the birational map BAR :  $\mathbb{K}^P \dashrightarrow \mathbb{K}^P$  given by BAR =  $\nabla \circ \Theta \circ \Delta^{-1}$  where  $(\Theta f)(x) = \frac{C}{f(x)}$ ,

## **Homomesy and Periodicity for BAR-motion**

On a general poset *P*, birational rowmotion usually has infinite order. However, surprisingly for certain "nice" posets such as  $[a] \times [b]$ , as well as types A and B positive root posets, birational rowmotion has the same small order as combinatorial rowmotion. Grinberg and Roby [GR14] proved that BOR<sup>*a+b*</sup> is the identity on the poset  $P = [a] \times [b]$ . Since BOR =  $\nabla^{-1} \circ$  BAR  $\circ \nabla$ , we have that BAR<sup>*a+b*</sup> is also the identity.

Let S be a collection of combinatorial objects, and  $f : S \to \mathbb{K}$  a "statistic" on S. We call f homomesic with respect to an invertible action  $\pi : S \to S$  if the (arithmetic) average of f over every  $\pi$ -orbit is the same [PR15]. A particular instance of homomesy with respect to  $\rho_A$  on the poset  $P = [a] \times [b]$  is in terms of fibers.

#### **Definition of Fibers in** $[a] \times [b]$

Fix  $a, b \in \mathbb{Z}^+$ . For  $1 \le k \le a$ , the subset  $\{(k, \ell) : 1 \le \ell \le b\}$  of  $[a] \times [b]$  is called the *k*th positive fiber. For  $1 \le \ell \le b$ , the subset  $\{(k, \ell) : 1 \le k \le a\}$  of  $[a] \times [b]$  is called the  $\ell$ th negative fiber.

#### Theorem (Propp–R. [PR15])

The statistics  $p_i : \mathcal{A}(P) \to \mathbb{Z}$  and  $n_i : \mathcal{A}(P) \to \mathbb{Z}$  where  $p_i(A)$  (resp.  $n_i(A)$ ) is 1 if A has an element in the *i*th positive fiber (resp. negative fiber) and 0 otherwise are homomesic with average b/(a+b) for  $p_i$  and a/(a+b) for  $n_i$  on any orbit. As the cardinality of an antichain can be expressed as  $p_1 + p_2 + \cdots + p_a$ , it follows that cardinality on  $\mathcal{A}(P)$  is homomesic with average ab/(a+b).

Each antichain  $A \in \mathcal{A}([a] \times [b])$  has an associated (a + b)-tuple w(A), called the Stanley–Thomas word. The proof of the above theorem relies on the fact that applying  $\rho_{\mathcal{A}}$  cyclically rotates the the Stanley–Thomas word given by

 $w_i = \begin{cases} 1 & \text{if } 1 \le i \le a \text{ and } A \text{ has an element in the } a \text{th positive fiber,} \\ 1 & \text{if } a + 1 \le i \le a + b \text{ and } A \text{ has NO element in the } (i - a) \text{th negative fiber,} \\ -1 & \text{otherwise.} \end{cases}$ 

**Orbits of**  $\rho_A$  **on**  $P = [2] \times [2]$  **and Stanley–Thomas Words** 

The symbol : Il means to repeat, so  $\rho_A$  has order 4. Below each labeling is its Stanley–Thomas word and cardinality.



 $(\nabla f)(x) = \frac{f(x)}{\sum\limits_{y \leqslant x} f(y)} \left( \text{with } f\left(\widehat{0}\right) = 1 \right),$   $(\Delta^{-1}f)(x) = \sum \left\{ f(y_1)f(y_2)\cdots f(y_k) : x = y_1 \lessdot y_2 \lessdot \cdots \lessdot y_k \lessdot \widehat{1} \right\}.$ 

In [GR14, EP18, MR19] the extension of order ideal rowmotion to Stanley's order polytope, and its birational lifting, have been well-studied. BAR-motion comes from the same philosophy, but starting with antichain rowmotion instead.





## **Birational Toggles**

Combinatorial rowmotion on order ideals or on antichains can be described either as the composition of three maps, or a composition of simple involutions called *toggles* [CF95, Str18, Jos19]. This lifts to the birational realm, and birational *order* toggles have been studied since around 2013. Birational *antichain* toggles are newly defined, as Einstein and Propp only studied BAR-motion in terms of transfer maps.

## **Birational Order Toggles [EP18, GR14]**

For  $e \in P$ , the **birational order toggle** at *e* is the birational map  $T_e : \mathbb{K}^P \dashrightarrow \mathbb{K}^P$  given by

$$(T_e(f))(x) = \begin{cases} f(x) & \text{if } x \neq e \\ \sum f(y) & \text{if } x \neq e \\ \frac{y \in \hat{P}, y \leq x}{f(e) \sum f(y)} & \text{if } x = e \end{cases}$$

where we set f(0) = 1 and f(1) = C.

#### **Birational Antichain Toggles (J.–R.)**

For  $e \in P$ , the **birational antichain toggle** at *e* is the rational map  $\tau_e : \mathbb{K}^P \dashrightarrow \mathbb{K}^P$  given by

$$\left(\tau_e(g)\right)(x) = \begin{cases} \frac{C}{\sum\limits_{(y_1,\dots,y_k)\in \mathrm{MC}_e(P)} g(y_1)\cdots g(y_k)} & \text{if } x = e\\ g(x) & \text{if } x \neq e \end{cases}$$

where  $MC_e(P)$  is the set of all maximal chains of *P* through *e*.



#### **Birational Stanley–Thomas Word**

ST word:

Let  $a, b \in \mathbb{Z}^+$ ,  $P = [a] \times [b]$ , and  $g \in \mathbb{K}^P$ . The **Stanley–Thomas word**  $ST_g$  is the (a + b)-tuple  $ST_g(i) = \begin{cases} g(i, 1)g(i, 2) \cdots g(i, b) & \text{if } 1 \le i \le a, \\ C/(g(1, i - a)g(2, i - a) \cdots g(a, i - a)) & \text{if } a + 1 \le i \le a + b. \end{cases}$ 

#### **BAR-motion Orbit on** $[2] \times [2]$



We have a birational analogue of fiber homomesy below. In the birational setting, to avoid dealing with taking *n*th roots, this manifests itself as certain products across an orbit equaling a fixed constant, independent of the initial labels.

## **Theorem (J.–R.): Fiber Homomesy for BAR-motion**

Let  $P = [a] \times [b]$ . For a labeling  $g \in \mathbb{K}^{P}$ ,  $\operatorname{ST}_{\operatorname{BAR}(g)}(i) = \operatorname{ST}_{g}(i-1)$  for  $2 \le i \le a+b$  and  $\operatorname{ST}_{\operatorname{BAR}(g)}(1) = \operatorname{ST}_{g}(a+b)$ . So,  $\prod_{m=0}^{a+b-1} (\operatorname{BAR}^{m}g)(k,1)(\operatorname{BAR}^{m}g)(k,2)\cdots(\operatorname{BAR}^{m}g)(k,b) = C^{b} \text{ for } 1 \le k \le a, \text{ and}$   $\prod_{m=0}^{a+b-1} (\operatorname{BAR}^{m}g)(1,\ell)(\operatorname{BAR}^{m}g)(2,\ell)\cdots(\operatorname{BAR}^{m}g)(a,\ell) = C^{a} \text{ for } 1 \le \ell \le b.$ 

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#### **Definition/Theorem [EP18]**

Let  $(x_1, x_2, ..., x_n)$  be any linear extension of *P*. The birational lift of order ideal rowmotion, herein called **birational order rowmotion (BOR-motion)**, is BOR =  $T_{x_1}T_{x_2} \cdots T_{x_n} = \Theta \circ \Delta^{-1} \circ \nabla$ .

#### **Theorem (J.–R.): BAR-motion is a Composition of Toggles**

Let  $(x_1, x_2, ..., x_n)$  be any linear extension of a finite poset *P*. Then BAR =  $\tau_{x_n} \cdots \tau_{x_2} \tau_{x_1}$ . Notice that the order of the poset elements in the composition of toggles is the opposite of that of BOR-motion (see left).

Example of BAR-motion as a Composition of Toggles  $x \xrightarrow{z} y \xrightarrow{\tau_{(1,1)}} x \xrightarrow{z} y \xrightarrow{\tau_{(2,1)}} y \xrightarrow{\tau_{(2,1)}} y \xrightarrow{w(x+y)} y \xrightarrow{\tau_{(2,2)}} y \xrightarrow{w(x+y)} y \xrightarrow{\tau_{(2,2)}} y \xrightarrow{w(x+y)} x \xrightarrow{w(x+y)} y \xrightarrow{\tau_{(2,2)}} y \xrightarrow{w(x+y)} x \xrightarrow{w(x+y)} y \xrightarrow{\tau_{(2,2)}} y \xrightarrow{w(x+y)} x \xrightarrow{w(x+y)} y \xrightarrow{\tau_{(2,2)}} y \xrightarrow{w(x+y)} y \xrightarrow{w(x+y)} y$ 

► To apply the toggle  $\tau_{(1,1)}$ , we consider the two maximal chains through (1,1).

►  $(1,1) \lt (2,1) \lt (2,2)$  with product of labels *wxz* 

►  $(1,1) \leq (1,2) \leq (2,2)$  with product of labels *wyz* 

Thus  $\tau_{(1,1)}$  changes the label at (1,1) to  $\frac{C}{wxz+wyz} = \frac{C}{w(x+y)z}$ .

Next we apply the toggle  $\tau_{(2,1)}$ . The only maximal chain through (2,1) is  $(1,1) \le (2,1) \le (2,2)$  with product of labels

 $\frac{C}{w(x+y)z}xz$ . Thus  $\tau_{(2,1)}$  changes the label at (2,1) to  $C / \left(\frac{C}{w(x+y)z}xz\right) = \frac{w(x+y)}{x}$ .

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The birational homomesy implies the corresponding result in the piecewise-linear realm. In the example orbit below of chain polytope rowmotion, note that the average "cardinality" (sum of labels) is  $\frac{2 \cdot 2}{2+2} = 1$ , as with antichain rowmotion.

$$0.1 \underbrace{\stackrel{0.3}{\underset{0.2}{\overset{0.4}{\overset{\rho_c}{\overset{}}}}}_{0.2} 0.4 \xrightarrow{\rho_c} 0.5 \underbrace{\stackrel{0.1}{\underset{0.1}{\overset{0.2}{\overset{}}}}_{0.1} 0.2 \xrightarrow{\rho_c} 0.1 \underbrace{\stackrel{0.2}{\underset{0.3}{\overset{0.4}{\overset{}}}}_{0.3} 0.4 \xrightarrow{\rho_c} 0.6 \underbrace{\stackrel{0.1}{\underset{0.1}{\overset{0.3}{\overset{}}}}_{0.1} 0.3 \xrightarrow{\rho_c} \bullet \blacksquare$$

label sum:0.4 + 0.6 = 10.6 + 0.3 = 0.90.3 + 0.7 = 10.7 + 0.4 = 1.1AVG:0.5 + 0.5 = 1

## Selected References (See abstract for more references and details.)

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Michael Joseph\* Tom Roby Dalton State College University of Connecticut Email: mjosephmath@gmail.com Email: tom.roby@uconn.edu