The hyperplane arrangement 
$$\mathcal{H}_{2n+1} = \{ x_i - x_j = y_k | 1 \leq i < j \leq n \} \subseteq \mathbb{R}^{2n}$$
was introduced by Hetyei in 2017. Using the finite field method of Athanasiadis, Hetyei showed that its number of regions is a median Genocchi number.

We refine Hetyei’s result by studying the intersection lattice \( \mathcal{L}(\mathcal{H}_{2n+1}) \) and its characteristic polynomial \( \chi_{\mathcal{L}(\mathcal{H}_{2n+1})}(t) \). By Zaslavsky’s formula, the number of regions of \( \mathcal{H}_{2n+1} \) is \( |\chi_{\mathcal{L}(\mathcal{H}_{2n+1})}(-1)| \).

We start our study by showing that \( \mathcal{L}(\mathcal{H}_{2n+1}) \) is an induced subposet of the lattice of partitions of \([2n]\). Theorem (L.-Wachs):

$$\sum_{n \geq 2} \chi_{\mathcal{L}(\mathcal{H}_{2n+1})}(t)x^n = \sum_{n \geq 2} \frac{(t-1)_n(1-t)_{n-1}x^n}{\prod_{k=1}^{n}(1-k(t-k)x)}$$

where \((a)_n\) is the falling factorial \( a(a-1) \cdots (a-n+1) \).

The proof constructs a bijection from the NBC sets of \( \mathcal{L}(\mathcal{H}_{2n+1}) \) to a class of permutations we call D-permutations (which are discussed below), and from there to a class of excedent functions known as surjective pistols.

When \( t = -1 \) and \( t = 0 \) yields the right-hand sides of the following formulas of Basky and Dumitriu (1981):

$$\sum_{n \geq 2} h_n x^n = \sum_{n \geq 2} \frac{n!}{\prod_{k=1}^{n}(1-k^2)x^n}$$

$$\sum_{n \geq 2} g_n x^n = \sum_{n \geq 2} \frac{n!}{\prod_{k=1}^{n}(1+k^2)x^n}$$

Hence, \( \chi_{\mathcal{L}(\mathcal{H}_{2n+1})}(0) = -g_n \) and \( \chi_{\mathcal{L}(\mathcal{H}_{2n+1})}(-1) = -h_n \).

Corollaries:

- (Hetyei) The number of regions of \( \mathcal{H}_{2n+1} \) is \( h_n \).
- (L.-Wachs) The Möbius invariant \( \mu(\mathcal{L}(\mathcal{H}_{2n+1})) \) of \( \mathcal{L}(\mathcal{H}_{2n+1}) \) is \(-g_n\).

D-Permutations

A D-permutation is a permutation \( \sigma \) satisfying, for all \( i \),

$$\sigma(2i) \leq 2i, \quad \sigma(2i-1) \geq 2i-1.$$

Theorem (L.-Wachs): The coefficient of \( t^{k-1} \) in \( \chi_{\mathcal{L}(\mathcal{H}_{2n+1})}(t) \) is \((-1)^k \) times the number of D-permutations on \([2n]\) with exactly \( k \) cycles.

Corollary (L.-Wachs):

- \#\{regions of \( \mathcal{H}_{2n+1} \)\} is the number of D-permutations on \([2n]\).
- \( \mu(\mathcal{L}(\mathcal{H}_{2n+1})) \) is \(-1\) times the number of D-cycles on \([2n]\).