Möbius Inversion as Duality

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Abstract

Many monoids and comonoids of interest in combinatorics can be equipped with a compatible poset structure.

In this poster, we show that for these monoids and comonoids, duality is best understood by introducing a second basis using Möbius inversion. We then use this to calculate primitives for poset comonoids.

Poset Monoids and Comonoids

A **poset species** is a map $\mathbf{H}$ that assigns

- to each finite set I a poset $\mathbf{H}[I]$
- to each bijection of finite sets $f : I \to J$ an order-preserving map $\mathbf{H}[I] \mapsto \mathbf{H}[J]$.

A **poset monoid** is a poset species $\mathbf{F}$ equipped with an order-preserving multiplication

$$m_{S,T} : \mathbf{F}[S] \times \mathbf{F}[T] \to \mathbf{F}[S \sqcup T]$$

for all disjoint finite sets S and T satisfying associativity and unitality axioms.

A **poset comonoid** is a poset species $\mathbf{F}$ equipped with an order-preserving comultiplication

$$\Delta_{S,T} : \mathbf{F}[S \sqcup T] \to \mathbf{F}[S] \times \mathbf{F}[T]$$

for all disjoint finite sets S and T satisfying coassociativity and counitality axioms.

Example 1: Let $\mathbf{SP}$ be the poset species with

$$\mathbf{SP}[I] = \{\text{Set Partitions of } I\},$$

where $\pi \leq \tau$ if and only if every part of π is a union of parts of τ.

For example, we have

$$125|346 \leq 15|3|2|46.$$

This is a poset monoid with multiplication

$$m_{S,T}(\pi, \tau) = \pi \sqcup \tau$$

and poset comonoid with comultiplication

$$\Delta_{S,T}(\pi) = (\pi|_S, \pi|_T).$$

Example 2: Let $\mathbf{G}$ be the poset species with

$$\mathbf{G} = \{\text{Graphs on vertex set } I\},$$

where $H \leq G$ if $E(H) \subset E(G)$.

This has two poset monoid structures. The first is given by

$$m_{S,T}(G, H) = G \sqcup H$$

and the second by

$$G \,\square_{S,T}\, H = (G^C \sqcup H^C)^C,$$

where G^c is the complement graph of G. That is, you add all edges between G and H.

This is a poset comonoid with comultiplication

$$\Delta_{S,T}(G) = (G|_S, G|_T).$$

Möbius Basis

Let $\mathbf{k}$ be a field of characteristic 0. Given a poset species $\mathbf{F}$, define its **linearization** $\mathbf{kF}$ by

$$\mathbf{kF}[I] = \text{free } \mathbf{k}\text{-vector space on } \mathbf{F}[I].$$

This has a standard basis given by the elements of $\mathbf{F}[I]$.

We can extend a (co)multiplication map linearly to get a **linearized (co)monoid**.

We define the **Möbius basis** of $\mathbf{kF}[I]$ to be the set $\{\omega_x \mid x \in \mathbf{F}[I]\}$ where

$$\omega_x = \sum_{\substack{y \in \mathbf{F}[I] \\ x \leq y}} \mu(x, y)\, y.$$

Adjoint Pairs

Let $(\mathbf{F}, \Delta)$ be a poset comonoid and $(\mathbf{F}, \square)$ be a poset monoid on the same poset species $\mathbf{F}$.

This comonoid and monoid are an **adjoint pair**, denoted by $\Delta \dashv \square$ if

$$\Delta_{S,T}(x) \leq (y, z) \iff x \leq y \,\square_{S,T}\, z,$$

for all disjoint finite sets S and T and for all $x \in \mathbf{F}[S \sqcup T], y \in \mathbf{F}[S]$ and $z \in \mathbf{F}[T]$.

Example 1: For $\mathbf{SP}$, we have $\Delta \dashv m$. That is,

$$\pi|_S \leq \tau_1 \text{ and } \pi|_T \leq \tau_2 \iff \pi \leq \tau_1 \sqcup \tau_2,$$

Example 2: For $\mathbf{G}$, we have $\Delta \dashv \square$. That is,

$$H|_S \leq G_1 \text{ and } H|_T \leq G_2 \iff H \leq G_1 \square G_2.$$

Duality

Given a linearized poset species $\mathbf{kF}$, we can define its linear dual by

$$\mathbf{kF}^*[I] = \mathrm{Hom}(\mathbf{kF}[I], \mathbf{k}).$$

A general fact is that if $(\mathbf{kF}, \Delta)$ is a comonoid, then $(\mathbf{kF}^*, \Delta^*)$ with

$$\Delta^*_{S,T}(f, g)(x) = f \otimes g \circ \Delta_{S,T}(x)$$

is a monoid and if $(\mathbf{kF}, \square)$ is a monoid, then $(\mathbf{kF}^*, \square^*)$ with

$$\square^*_{S,T}(f)(x \otimes y) = f(x \,\square_{S,T}\, y),$$

is a comonoid.

Main Theorem

If $(\mathbf{F}, \Delta)$ and $(\mathbf{F}, \square)$ are an adjoint pair, then

$$(\mathbf{kF}, \square) \cong (\mathbf{kF}^*, \Delta^*)$$

through the map

$$x \mapsto \omega_x^*,$$

where ω_x^* is the function defined by $\omega_x^*(\omega_y) = \delta_{x,y}$.

Concretely,

$$\Delta_{S,T}(\omega_x) = \sum_{x_1 \square x_2 = x} \omega_{x_1} \otimes \omega_{x_2}.$$

Primitives

For a linearized comonoid $(\mathbf{kF}, \Delta)$ an element $x \in \mathbf{kF}[I]$ is **primitive** if

$$\Delta_{S,T}(x) = 0,$$

for all $S \sqcup T = I$ with S and T non-empty.

Question: Find a nice basis for the vector space of primitives of $\mathbf{kF}[I]$.

Corollary: If $(\mathbf{kF}, \Delta)$ and $(\mathbf{kF}, \square)$ are an adjoint pair, then a basis for the primitives of $\mathbf{kF}[I]$ is given by

$$\{\omega_x \mid x \text{ is } \square\text{-indecomposable.}\},$$

where x is $\square$-indecomposable if there is no S and T such that $x = x_1 \,\square_{S,T}\, x_2$ for some $x_1 \in \mathbf{F}[I]$ and $x_2 \in \mathbf{F}[I]$.

Example 1: For $\mathbf{SP}$, the only set partition of I that is m-indecomposable is the partition containing one part I. Thus, the primitives of $\mathbf{kSP}[I]$ are spanned by

$$\omega_I = \sum_\pi \mu(I, \pi)\pi.$$

General machinery and a result by Ardila and Aguiar [1] allows us to interpret this result in the ring of symmetric functions Sym. This recovers a classic result by Doubilet [2]:

$$p_n = \frac{1}{\mu(\hat{0}, \hat{1})} \sum_\pi \mu(\hat{0}, \pi) h_{\lambda(\pi)},$$

where $\lambda(\pi)$ is the integer partition of the sizes of the parts of π, p_n is the power sum basis of Sym, and h_λ is the homogeneous basis of Sym.

Example 2: For $\mathbf{G}$, a graph $G \in \mathbf{G}[I]$ is $\square$-indecomposable if and only if its complement G^c is connected. Thus, a basis for the primitives of $\mathbf{G}[I]$ is

$$\{\omega_G = \sum_{G \leq H} (-1)^{|E(H) - E(G)|} H \mid G^c \text{ is connected}\}.$$

Other Applications

- Calculate primitives for Hypergraphs, Simplicial Complexes, Generalized Permutahedra, Poset of Posets, Scheduling Problems, and Matroids.
- Prove cofreeness for the above comonoids.
- Expand various symmetric function invariants in the power-sum basis.

References

[1] M. Aguiar and F. Ardila.
Hopf monoids and Generalized Permutahedra.
ArXiv e-prints, September 2017.

[2] P. Doubilet.
On the Foundations of Combinatorial Theory. VII: Symmetric Functions through the Theory of Distribution and Occupancy.
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