Schubert Structure Operators

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Introduction

The cohomology ring of a flag variety comes with a preferred basis of "Schubert classes", and it is known for geometric reasons that products thereof expand positively in the basis. Outside of special cases (e.g. products of Grassmannian classes), there are no combinatorial formulas known or even conjectured for the structure constants.

We consider the *equivariant* cohomology of the flag variety G/B, and introduce new operators whose coefficients compute these (now polynomial) Schubert structure constants in a manifestly polynomial (but not positive) way, resulting in a symmetric formula generalizing the positive AJS/Billey formula (coefficient of S_w in $S_w S_v$).

Notation and background (and standard example)

G complex reductive Lie group;
$G = Gl(n, \mathbb{C})$, for example

$T \leq G$ maximal torus;
T diagonal matrices

B, B_- Borel and opposite Borel subgroups;
B, B_- upper & lower triangular matrices

$W = N(T)/T$ Weyl group;
$W = S_n$, the group of permutations

$\{\alpha_i\}$ simple roots and associated simple reflections $\{r_\alpha\}$;

G/B flag manifold, with left T-action;
$G/B = \{0 \subset V_1 \subset V_2 \subset \cdots \subset \mathbb{C}^n\}$ with $\dim V_i = i$, i.e. complete flags

$\{wB/B \ : \ w \in W\}$ T-fixed points;
fixed points correspond to permutations

H_T^* the T-equivariant cohomology of a point with coefficients in $\mathbb{Q}$;
H_T^* is a polynomial ring

$H_T^*(G/B)$, T-equivariant cohomology of G/B;
$H_T^*(G/B) = H_T^* \otimes_{(H_T^*)^W} H_T^*$

$\{S_v \in H_T^*(G/B) \ : v \in W\}$ Schubert classes;
S_v is Poincaré dual to $\overline{B\_vB}/B$.

$c_{uv}^w \in H_T^*$ structure constants defined in $H_T^*(G/B)$ by

$$S_u S_v = \sum_w c_{uv}^w S_w$$

Structure constant positivity. When c_{uv}^w is written (uniquely) as a sum of monomials in the $\{\alpha_i\}$, each monomial has a non-negative coefficient [Graham '01].

Divided difference operators.
The **nil Hecke algebra** has a $\mathbb{Z}$-basis $\{\partial_w \ : \ w \in W\}$, products defined by

$$\partial_w \partial_v := \begin{cases} \partial_{wv} & \text{if } \ell(vw) = \ell(v) + \ell(w) \\ 0 & \text{otherwise, i.e. if } \ell(vw) < \ell(v) + \ell(w). \end{cases}$$

The $\{\partial_w\}$ act on the ring H_T^*: for each root α, $\partial_{r_\alpha} \cdot f := (f - r_\alpha f)/\alpha$.

The nil Hecke algebra acts on the first factor of $H_T^* \otimes_{\mathbb{Z}} H_T^*$, and this action descends to the quotient $H_T^* \otimes_{(H_T^*)^W} H_T^* \cong H_T^*(G/B)$.

Schubert structure operators

Let $H_T^*[\partial]$ denote the smash product of H_T^* with $\mathbb{Z}[\partial]$, the algebra consisting of the free H_T^*-module $H_T^* \otimes_{\mathbb{Z}} \mathbb{Z}[\partial]$ with product given by, for $p, q \in H_T^*$,

$$(p \otimes \partial_v) \cdot (q \otimes \partial_w) = p(\partial_v q) \otimes \partial_v \partial_w$$

and extended linearly. Note r_α acts on $H_T^*(G/B)$ equivalently to $1 - \alpha\partial_\alpha$.

Schubert structure operators: We introduce $K^\alpha \in H_T^*[\partial] \otimes \mathbb{Z}[\partial] \otimes \mathbb{Z}[\partial]$,

$$K^\alpha := (\partial_\alpha r_\alpha \otimes 1 \otimes 1) + (r_\alpha \otimes \partial_\alpha \otimes 1) + (r_\alpha \otimes 1 \otimes \partial_\alpha) + (\alpha\, r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha).$$

Theorems

1. The $\{K^\alpha\}$ satisfy commuting and braid relations in simply- and doubly-laced types (and, we conjecture, in the the G_2 case as well):
   - Each $(K^\alpha)^2 = 0$, and $K^\alpha K^\beta = K^\beta K^\alpha$ if α and β are not connected.
   - $K^\alpha K^\beta K^\alpha = K^\beta K^\alpha K^\beta$ or $K^\alpha K^\beta K^\alpha K^\beta = K^\beta K^\alpha K^\beta K^\alpha$ if α and β are connected by 1 or 2 bonds, respectively.
   - Thus we may define $K^w := \prod_{q \in Q} K^q$, for Q any reduced word for w.
2. $$K^w(S_1 \otimes S^1 \otimes S^1) = \sum_{u,v} c_{uv}^w \otimes S^u \otimes S^v$$
   where S_v are Schubert classes and S^v are opposite Schubert classes.
3. (Restatement of #2) Let Q be a reduced word for w. Then

$$c_{uv}^w = \sum_{\substack{P,R \subseteq Q \text{ reduced} \\ \prod P = u,\ \prod R = v}} \prod_Q \left(\alpha_q{}^{[q \in P,R]}\, \partial_q{}^{[q \notin P,R]}\, r_q \right) \cdot 1$$

where the exponent "$[\sigma]$" is 1 if the statement σ is true, 0 if false.

Computing structure constants: examples

1. Let $Q = 12312$, so $w = r_1 r_2 r_3 r_1 r_2 = [3421]$ in one-line notation, and take $u = r_2 r_3 r_2 = [1432]$, $v = r_1 r_2 r_1 = [3214]$. Then $P = -23-2$ and $R \in \{12-1-, -2-12\}$ as subwords of Q so we have

$$\begin{aligned} c_{uv}^w &= (r_1\ \alpha_2 r_2\ r_3\ r_1\ r_2 + \partial_1 r_1\ \alpha_2 r_2\ r_3\ r_1\ \alpha_2 r_2) \cdot 1 \\ &= (\alpha_1 + \alpha_2) \cdot 1 + \partial_1(\alpha_1 + \alpha_2)(\alpha_2 + \alpha_3) \cdot 1 \\ &= \alpha_1 + \alpha_2 + \partial_1(\alpha_1 + \alpha_2)\alpha_2 \cdot 1 + \partial_1(\alpha_1 + \alpha_2)\alpha_3 \cdot 1 \ = \ \alpha_1 + \alpha_2 + \alpha_3. \end{aligned}$$

2. If $u = w$, then P must be all of Q, and the formula simplifies to one equivalent to AJS/Billey (computing the point restriction $S_v|_w = c_{wv}^w$):

$$c_{wv}^w = \sum_{\substack{R \subseteq Q \text{ reduced} \\ \prod R = v}} \prod_Q \left(\alpha_q{}^{[q \in R]}\, r_q \right) \cdot 1$$

Bott-Samelson manifolds, used in the proof

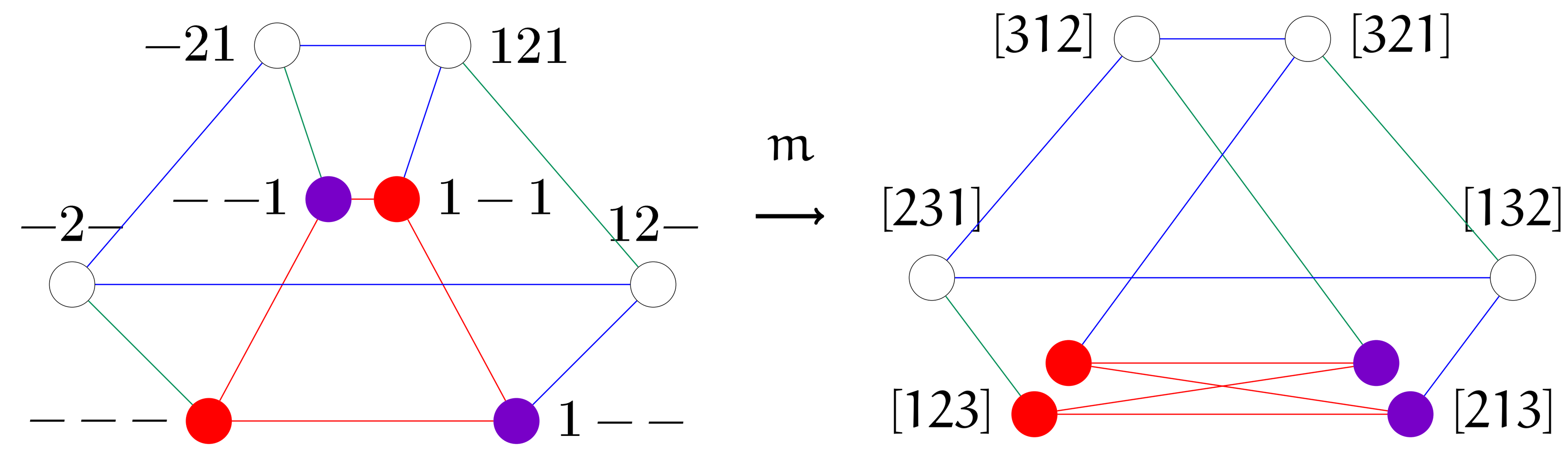
Choose a reduced word $Q = r_{\alpha_{i_1}} r_{\alpha_{i_2}} \cdots r_{\alpha_{i_\ell}}$ for $w \in W$.
The associated **Bott-Samelson manifold** and **map** are

$$BS^Q = P_{\alpha_{i_1}} \times^B P_{\alpha_{i_2}} \times^B \cdots \times^B P_{\alpha_{i_\ell}}/B \quad \xrightarrow{m} \quad G/B$$

with $P_{\alpha_{i_j}}$ the minimal parabolic associated to the simple reflection r_{i_j}, where the equivalence of elements is

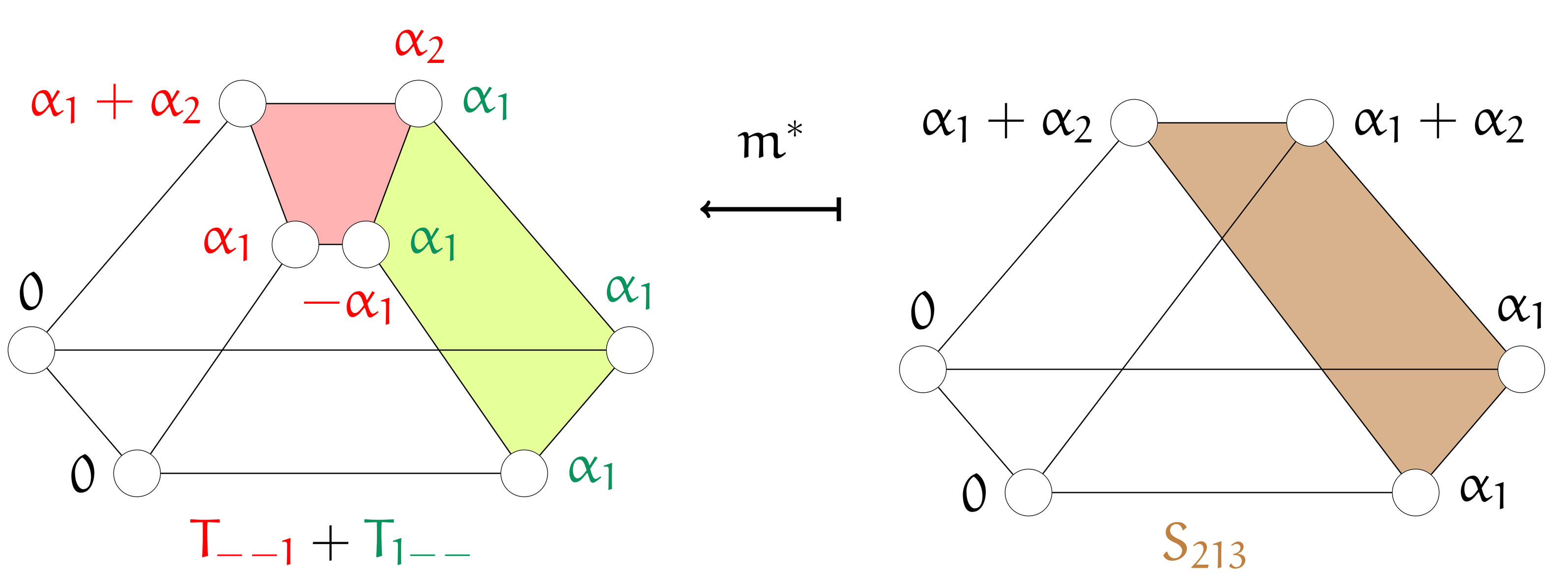
$$(g_1, g_2, \ldots, g_\ell) \sim (g_1 b_1, b_1^{-1} g_2 b_2, \ldots, b_{\ell-1}^{-1} g_\ell b_\ell) \qquad \mapsto \qquad g_1 g_2 \cdots g_\ell B/B.$$

Its T-fixed points form the vertices 2^Q of a cube, which map via m to W.



Each $R \subseteq Q$ gives a submanifold $BS^R \hookrightarrow BS^Q$, together a basis $\{[BS^R]\}$ of $H_*^T(BS^Q)$, with dual basis $\{T_R = \prod_{r \in R} T_r\}$ of $H_T^*(BS^Q)$. The key lemmas are

$$m^*(S_w) = \sum_{\substack{R \subseteq Q \text{ reduced} \\ \prod R = w}} T_R \qquad \boxed{T_R T_S = \sum_{\substack{J \subseteq Q \\ J \supseteq R,S}} T_J \prod_{q \in J} \left(\alpha_q{}^{[q \in R,S]}\, \partial_q{}^{[q \notin R,S]}\, r_q \right) \cdot 1}$$



Corollaries (two recursive formulæ)

Fix a reflection r_α, and let $\overline{s}$ denote $r_\alpha s$ for $s \in W$. If $\overline{w} < w$, then

$$c_{uv}^w = (\partial_\alpha r_\alpha) \cdot c_{uv}^{\overline{w}} + [\overline{u} < u] c_{\overline{u},v}^{\overline{w}} + [\overline{v} < v] c_{u,\overline{v}}^{\overline{w}} + [\overline{u} < u][\overline{v} < v] \alpha c_{\overline{u},\overline{v}}^{\overline{w}}$$

where $[\overline{s} < s]$ indicates 1 if $\overline{s} < s$, and 0 otherwise (i.e. $\overline{s} > s$).

Similarly, let $\underline{s}$ denote $s r_\alpha$. If $\underline{w} < w$, then

$$c_{uv}^w = [\underline{u} < u] \left(c_{\underline{u},v}^{\underline{w}}\right) + [\underline{v} < v] \left(c_{u,\underline{v}}^{\underline{w}}\right) + [\underline{u} < u][\underline{v} < v] \left(d_{\underline{u},\underline{v}}^{\underline{w}} \cdot \alpha\right)$$

Future directions

- Find an interpretation of K^w applied to other triple products of Schubert (and opposite Schubert) classes, not just to $S_1 \otimes S^1 \otimes S^1$.
  Or pretty much equivalently, find a closed form for K^w.
- Find a purely algebraic proof of #2 and #3, not relying on Bott-Samelsons.
- Extend to K-theory and CSM classes.
- Extend to the pullback $H^*(G/B) \to H^*(K/B_K)$, K a symmetric subgroup.

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