The Hopf monoid of orbit polytopes

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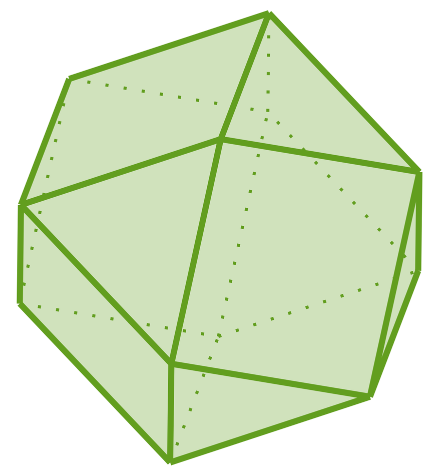


Introduction

Many families of combinatorial objects can be endowed with an algebraic structure called a Hopf monoid. The collection of multiplicative functions defined on a Hopf monoid forms the character group. Aguiar and Ardila described the character groups of the Hopf monoids of permutahedra and associahedra [1]. This project introduces the Hopf monoid OP of orbit polytopes, which are the generalized permutahedra invariant under the action of the symmetric group. The Hopf structure of OP can be described using integer compositions, and the character group is related to *NSym*, the Hopf algebra of noncommutative symmetric functions.

Hopf Monoids

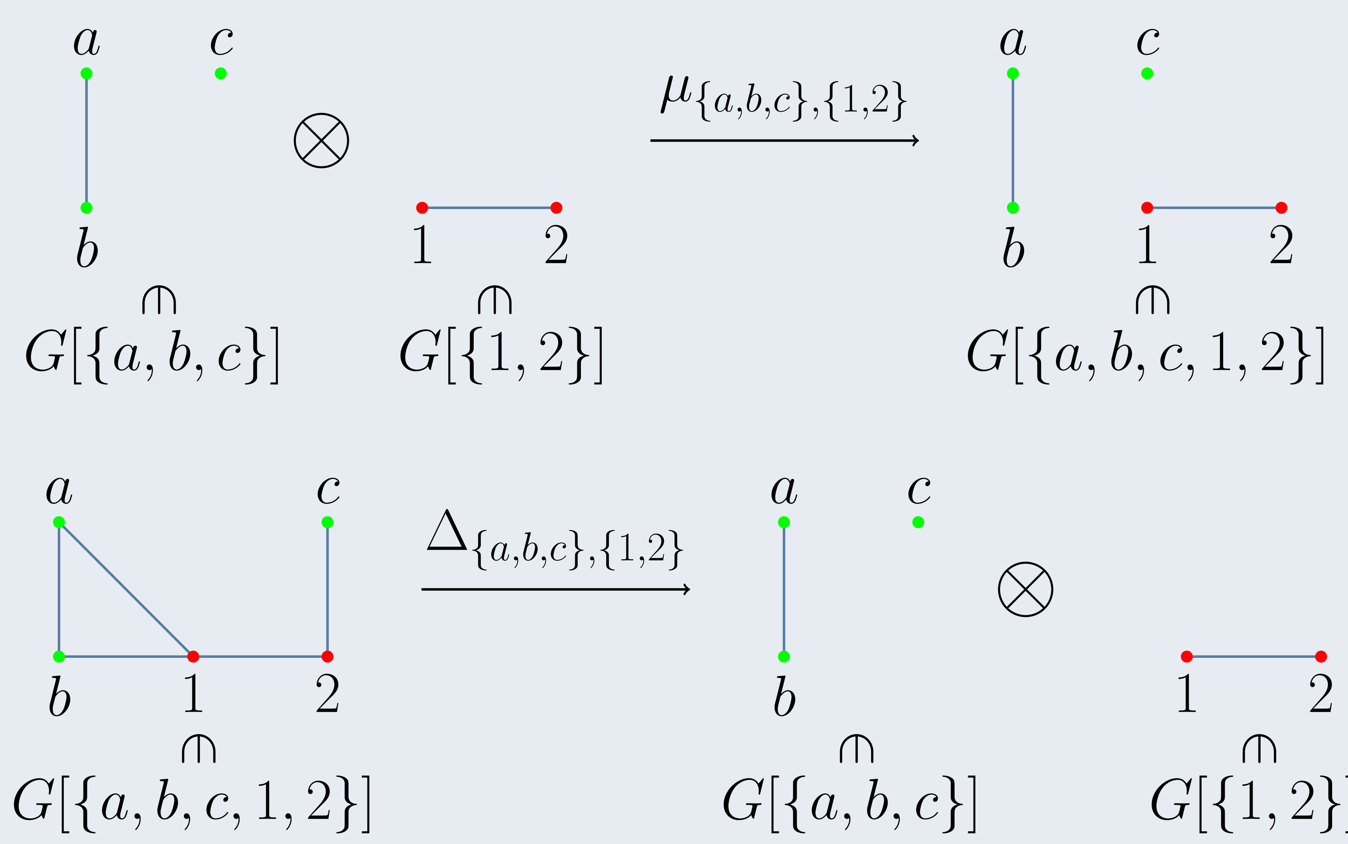
A **Hopf monoid** is a vector species H along with **product** and **coproduct** maps

$$\mathrm{H}[S] \otimes \mathrm{H}[T] \xrightarrow{\mu_{S,T}} \mathrm{H}[I] \quad \text{and} \quad \mathrm{H}[I] \xrightarrow{\Delta_{S,T}} \mathrm{H}[S] \otimes \mathrm{H}[T]$$

for each ordered set partition $I = S \sqcup T$. These maps must satisfy *naturality*, *unitality*, *associativity*, and *compatibility* axioms.

Example: Hopf monoid of graphs [1]

Let G be the Hopf monoid of graphs. The product is given by disjoint union, and the coproduct is given by induced subgraphs.



The Hopf Monoid of Orbit Polytopes

Let $\mathbb{R}^I$ be the linearization of the set I. The **orbit polytope $\mathcal{O}(p)$** is the convex hull of all permutations of the coordinates of the point $p \in \mathbb{R}^I$. Orbit polytopes live in a codimension-1 subspace of $\mathbb{R}^I$.

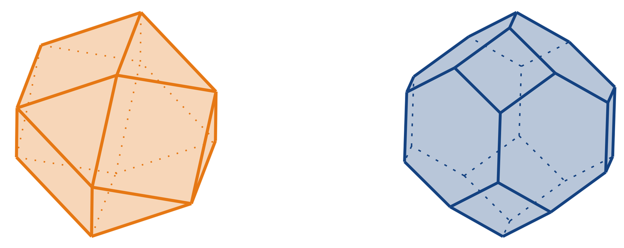


Figure: Orbit polytopes $\mathcal{O}(2,1,1,0)$ and $\mathcal{O}(3,2,1,0)$ in $\mathbb{R}^{[4]}$.

Up to normal equivalence, orbit polytopes in $\mathbb{R}^I$ are in bijection with compositions of the integer $|I|$. The equivalence class of polytopes labelled by the composition α is called $\mathcal{O}_\alpha$.

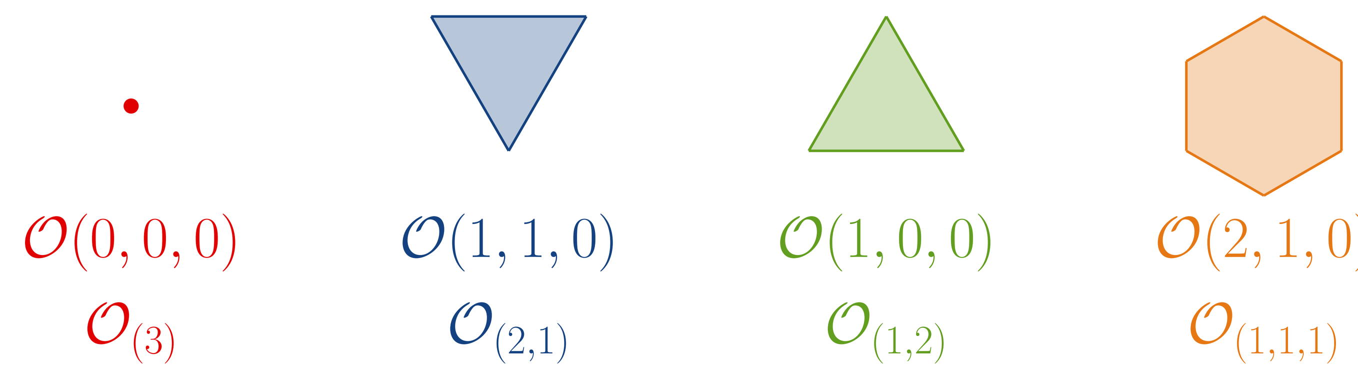


Figure: Orbit polytopes in $\mathbb{R}^{[3]}$ are in bijection with compositions of 3.

The **product** of polytopes $P \subset \mathbb{R}^S$ and $Q \subset \mathbb{R}^T$ is the direct product of P and Q as sets:

$$P \cdot Q := \{(p,q) : p \in P, q \in Q\} \subset \mathbb{R}^{S \sqcup T}$$

Define a **coproduct** on orbit polytopes: To compute $\Delta_{S,T}(P)$, find the face of P maximizing the functional $1_S := \sum_{s \in S} e_s$. This face decomposes into a product of lower dimensional orbit polytopes.

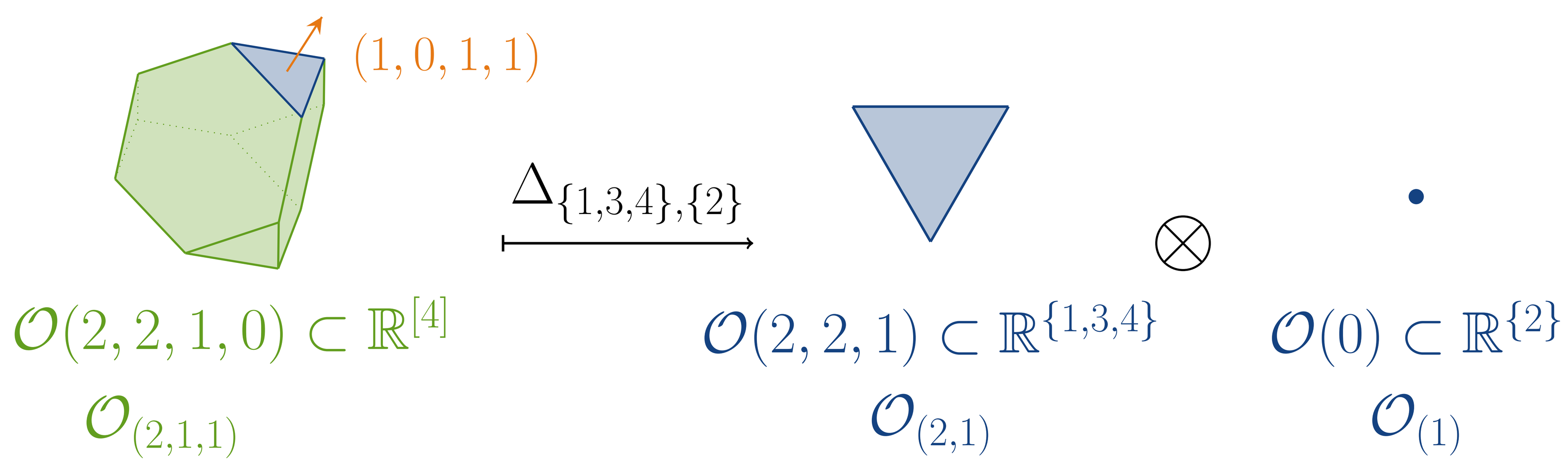


Figure: The coproduct $\Delta_{\{1,3,4\},\{2\}}(\mathcal{O}(2,2,1,0))$.

This product and coproduct behave nicely with normal equivalence, which lets us define a **Hopf monoid OP of orbit polytopes**.

Describing the Character Group of OP

For a field $\Bbbk$, a **character** on a Hopf monoid H is a collection of multiplicative maps $\zeta = \{\zeta_I : \mathrm{H}[I] \to \Bbbk\}$, one for each finite set I. Define a convolution product $*$ on characters:

$$(\zeta * \psi)_I(x) = \sum_{I = S \sqcup T} \mathrm{mult}_\Bbbk \circ (\zeta_S \otimes \psi_T) \circ \Delta_{S,T}(x)$$

The **character group** is the set of characters equipped with $*$.

Main theorem [3]

The character group of OP is isomorphic to a subgroup of the group of invertible elements in the completion of $NSym$, the Hopf algebra of noncommutative symmetric functions.

To describe the isomorphism, we need two operations on compositions:

- **Concatenation**:
$$\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell)$$
- **Near-concatenation** [2]:
$$\alpha \odot \beta = (\alpha_1, \ldots, \alpha_k + \beta_1, \ldots, \beta_\ell)$$

The isomorphism sends a character ζ on OP to the generating function

$$\sum_{\beta \in \mathrm{Comp}} \frac{\zeta(\mathcal{O}_\beta)}{|\beta|!} R_\beta,$$

where $\{R_\beta\}$ is the noncommutative ribbon basis of $NSym$. This means that $R_\alpha R_\beta = R_{\alpha \cdot \beta} + R_{\alpha \odot \beta}$.

Acknowledgements

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References

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[2] D. Grinberg and V. Reiner, "Hopf algebras in combinatorics," arXiv:1409.8356v5 [math.CO], May 2018.

[3] M. Supina, "The Hopf monoid of orbit polytopes," arXiv:1904.08437 [math.CO], Apr. 2019.