Snake Graphs from Orbifolds

Esther Banaian[1†] Elizabeth Kelley[1†]

Snake Graphs

A snake graph is a labeled collection of square tiles, glued along their north or east edges. Snake graphs can be used to encode the Laurent expansions of cluster variables in cluster algebras of surface type [5].

Snake Graph Formula [5]

Let (S, M) be a bordered surface with triangulation T, $\mathcal{A}$ be the corresponding cluster algebra with principal coefficients, and γ be an ordinary arc on S. Then x_γ can be written as

$$x_\gamma = \frac{1}{\text{cross}(T,\gamma)} \sum_P x(P) y(P)$$

where P is a perfect matching of $G_{T,\gamma}$ and

$$\text{cross}(T,\gamma) := x_{i_1} \cdots x_{i_d} \text{ for } \tau_{i_1}, \ldots, \tau_{i_d} \text{ crossed by } \gamma$$
$$x(P) := x_{i_1} \cdots x_{i_k} \text{ for } \tau_{i_1}, \ldots, \tau_{i_k} \text{ labeling edges in } P$$
$$y(P) := \prod_{i=1}^{n} h_{\tau_i}^{m_i}$$

where m_i is the multiplicity of τ_i in $P \ominus P_-$ and $h_{\tau_i} = y_{\tau_k}$ unless τ_i is an edge of a self-folded triangle.

Generalized Cluster Algebras

Fix a semifield $(\mathbb{P}, \oplus, \cdot)$ and let $F = \mathbb{QP}[x_1, \ldots, x_n]$. A *generalized cluster seed* in F is a quadruple $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ where $\mathbf{x}$, $\mathbf{y}$, and B are defined as in ordinary cluster algebras and $\mathbf{Z}$ is a collection of exchange polynomials

$$Z_i(u) = z_{i,0} + z_{i,1}u + \cdots + z_{i,d_i}u^{d_i}$$

with all $z_{i,j} \in \mathbb{P}$ and $z_{i,0} = z_{i,d_i} = 1$.

Generalized cluster algebras with all $d_i \in \{1, 2\}$ can be modeled as a triangulated orbifolds via the dictionary:

$$\text{initial generalized cluster seed} \leftrightarrow \text{initial triangulation}$$
$$\text{other cluster variables} \leftrightarrow \text{other arcs on the orbifold}$$
$$\text{mutation } \mu_k \leftrightarrow \text{“flipping” arc } \tau_k$$

The cluster variable x_i corresponds to an *ordinary arc* if $d_i = 1$ and to a *pending arc* if $d_i = 2$.

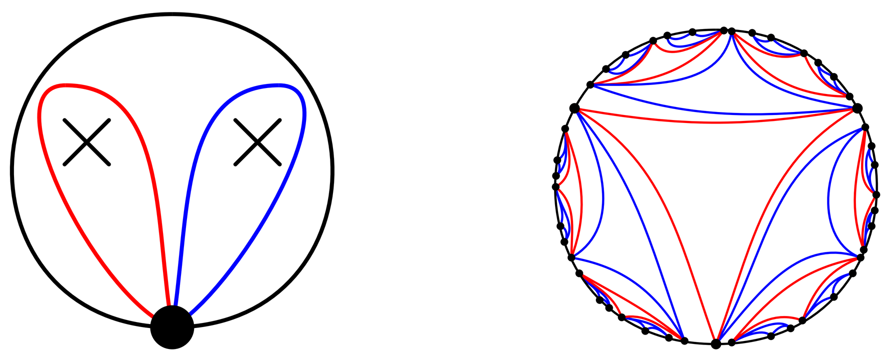


Figure: A surface with two orbifold points and a covering space when both points are order 3.

Orbifolds

An orbifold is a generalization of a manifold where the local structure is given by quotients of open subsets of $\mathbb{R}^n$ under finite group actions.
Each orbifold point, denoted as $\times$, has associated integer order p. Intuitively, an orbifold point of order p is "$1/p^{\text{th}}$" of a point. A winding arc with k self-intersections "sees" the orbifold point as a puncture if $k < p$ and as an ordinary point if $k = p$.

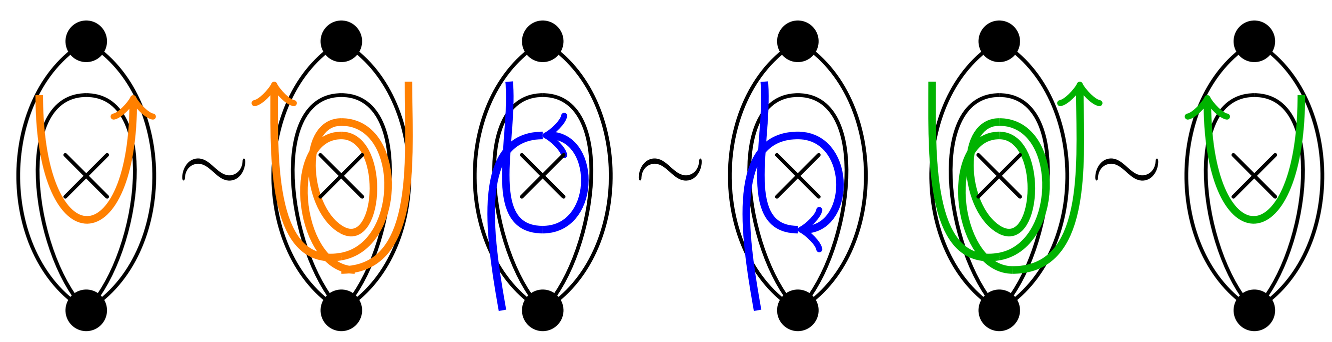
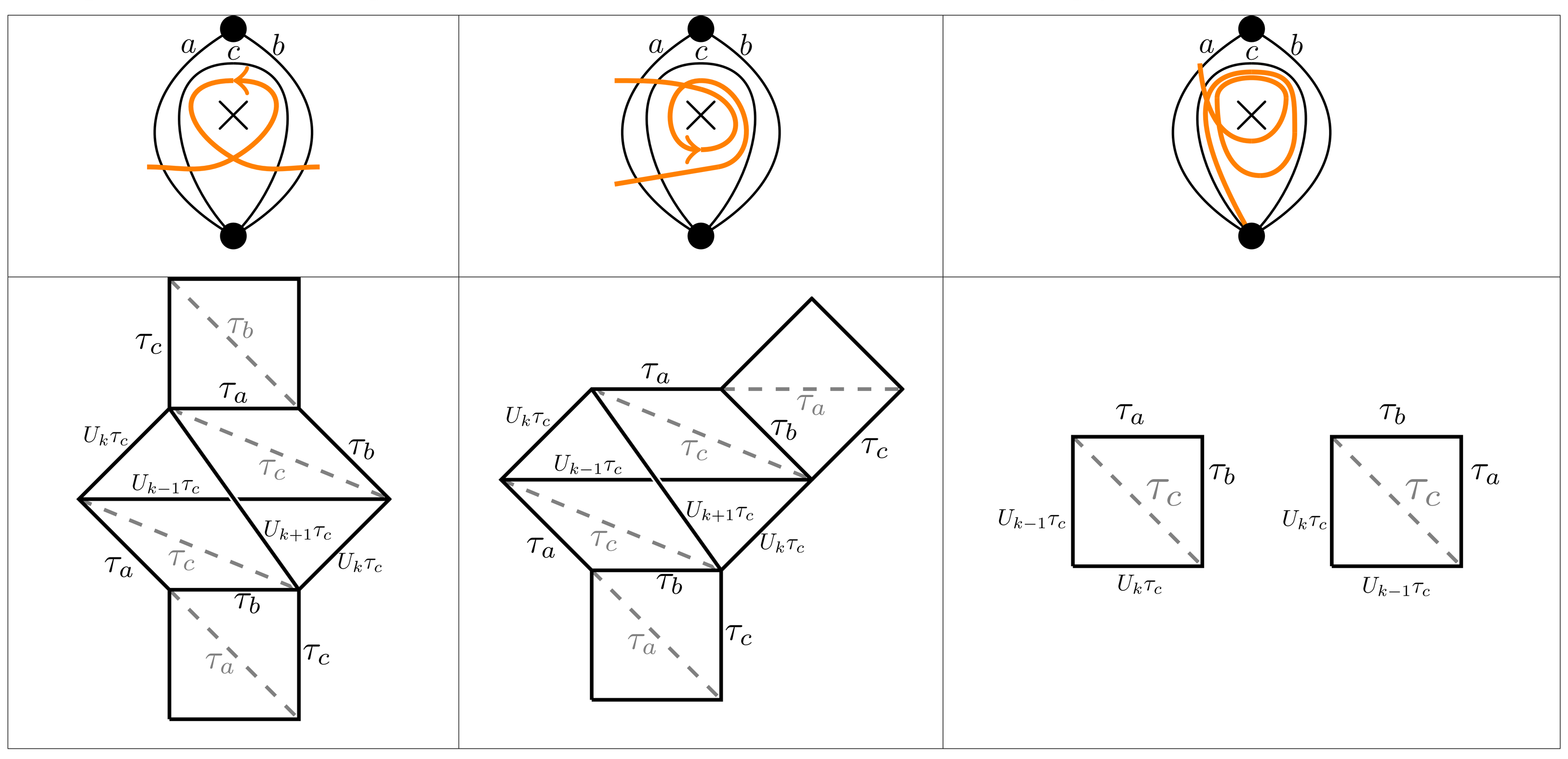


Figure: Arc winding behavior around an orbifold point of order $p = 4$.

Snake Graphs from Orbifolds

Snake graphs from triangulated orbifolds are built from the following puzzle pieces:



Extension of Snake Graph Formula

Using this construction, the MSW snake graph formula also holds for ordinary and *generalized arcs* on an orbifold surface, $\mathcal{O} = (S, M, Q)$. *Good matchings* of *band graphs* also encode Laurent expansions of closed curves.

Remark

We let U_k denote the k-th normalized Chebyshev **evaluated at** $\lambda_p = 2\cos(\pi/p)$ where p is the order of the relevant orbifold point.
The first few values of λ_p are

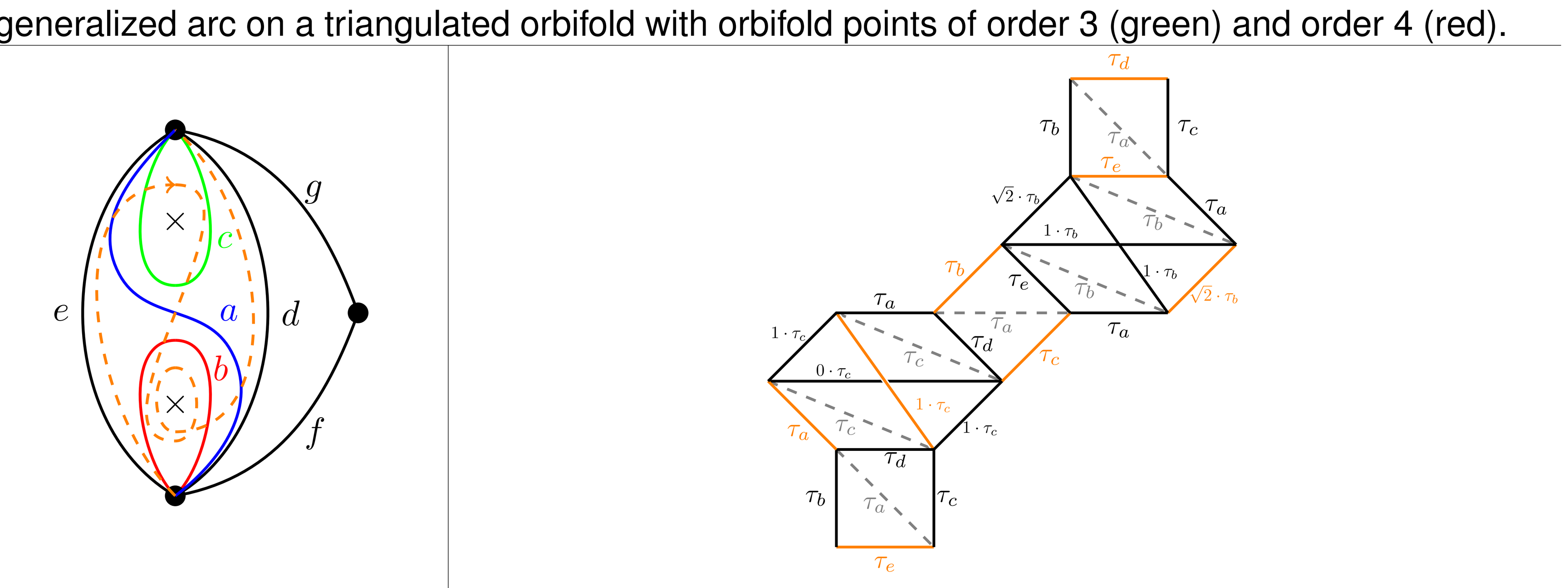
$$\lambda_3 = 1, \lambda_4 = \sqrt{2}, \lambda_5 = \frac{1+\sqrt{5}}{2}, \lambda_6 = \sqrt{3}.$$

The polynomials $U_k(x)$ are defined by the following for all $k \geqslant 1$:

$$U_{-1}(x) = 0, U_0(x) = 1$$
$$U_k(x) = xU_{k-1}(x) - U_{k-2}(x)$$

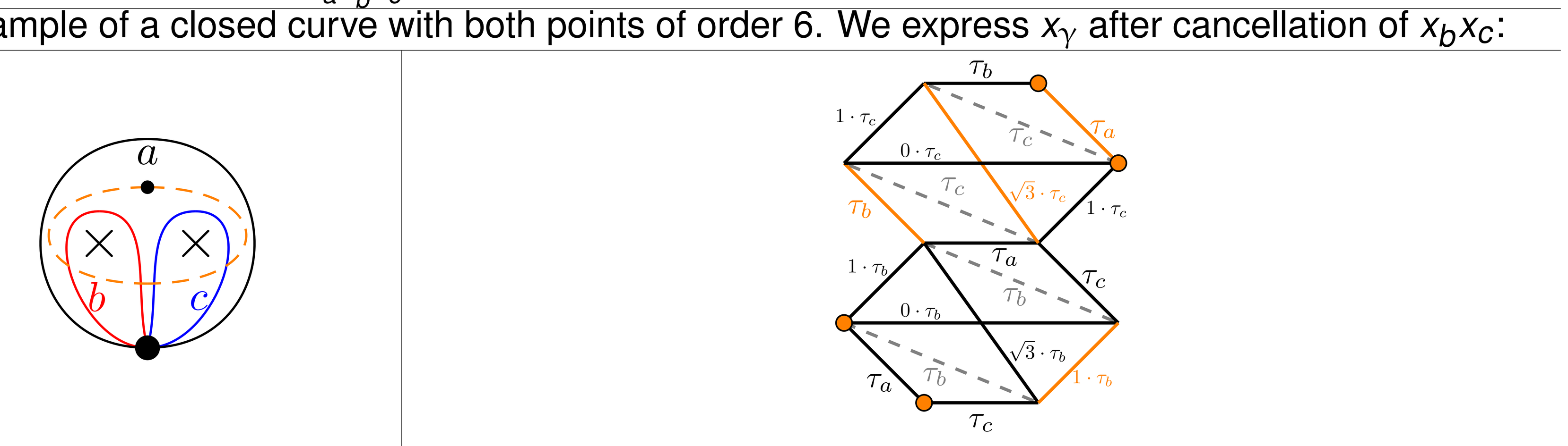
Examples of Snake Graphs

A generalized arc on a triangulated orbifold with orbifold points of order 3 (green) and order 4 (red).



$$x_\gamma = \frac{1}{x_a^3 x_b^2 x_c^2}\left(\sqrt{2} y_a y_b^2 y_c x_a x_b^2 x_c^3 x_d x_e^2 + y_a y_b y_c^2 x_a x_b^2 x_c^2 x_d^2 x_e + \cdots \right.$$

An example of a closed curve with both points of order 6. We express x_γ after cancellation of $x_b x_c$:



$$x_\gamma = \frac{1}{x_b x_c}(x_b^2 + \sqrt{3} y_c x_a x_b + y_c^2 x_a^2 + \sqrt{3} y_b y_c^2 x_a x_c + y_b^2 y_c^2 x_c^2)$$

Orbifold Frieze Patterns

A frieze pattern is an array of infinite rows of numbers which satisfy a certain local arithmetic property. A particular frieze pattern is uniquely determined by this arithmetic property and its *quiddity row*, the first non-trivial row.

We can define the quiddity sequence associated to a triangulated orbifold by taking the concatenation of each two consecutive boundary components, computing their Laurent expansion, and then specializing all variables to one. This will uniquely determine a frieze pattern whose entries are specializations of elements of the generalized cluster algebra.

These frieze patterns are finite if and only if the orbifold surface has zero or one orbifold points and is homeomorphic to a disk with marked points on the boundary.

Infinite frieze patterns have a family of invariants called *growth coefficients*. As in the surface case, our band graphs encode the growth coefficients.

Example of Frieze Patterns

The following frieze pattern corresponds to our first triangulation in the main example.

0		0		0		0	
	1		1		1		1
$4+\sqrt{2}$		1		5		$4+\sqrt{2}$	
	$3+\sqrt{2}$		4		$19+5\sqrt{2}$		$3+\sqrt{2}$
$14+5\sqrt{2}$		$11+4\sqrt{2}$		$15+4\sqrt{2}$		$14+5\sqrt{2}$	
	$51+20\sqrt{2}$		$49+26\sqrt{2}$		$11+4\sqrt{2}$		$51+20\sqrt{2}$
			⋮				

Future Directions

- Can this construction be extended to generalized cluster algebras that don't correspond to triangulated orbifolds?
- Do our snake graphs help describe algebraic structure in a polygon-dissected surface?
- Is there an elementary way to classify the frieze patterns we can construct in this manner?

References

[1] Karin Baur, Klemens Fellner, Mark J. Parsons, and Manuela Tschabold. "Growth Behaviour of Periodic Tame Friezes".
[2] Leonid Chekhov and Michael Shapiro. "Teichmüller Spaces of Riemann Surfaces with Orbifold Points of Arbitrary Order and Cluster Variables". In: *International Mathematics Research Notices* 2014.10 (2014), pp. 2746-2772.
[3] Anna Felikson, Michael Shapiro and Pavel Tumarkin. "Cluster Algebras and Triangulated Orbifolds". In *Advances in Mathematics* 231.5 (2012), pp. 2953-3002.
[4] Sergey Fomin and Dylan Thurston. "Cluster Algebras and Triangulated Surfaces II: Lambda Lengths". In *Memoirs of the American Mathematical Society*(2012).
[5] Sergey Fomin and Andrei Zelevinsky. "Cluster Algebras I: Foundations". In *Journal of the American Mathematical Society* 15.2 (2002), pp. 497-529.
[6] Gregg Musiker, Ralf Schiffler, and Lauren Williams. "Positivity for Cluster Algebras from Surfaces". In *Advances in Mathematics* 227.6 (2011), pp. 2241-2308.

[1]University of Minnesota, banai003@umn.edu & kell1642@umn.edu

This research was conducted at the University of Minnesota, supported by NSF Grant DMS- 1745638