**Snake Graphs**

A snake graph is a labeled collection of square tiles, glued along their north or east edges. Snake graphs can be used to encode the Laurent expansions of cluster variables in cluster algebras of surface type [5].

**Snake Graph Formula [5]**

Let \((S, M)\) be a bordered surface with triangulation \(T\), \(x_i\) be the corresponding cluster algebra with principal coefficients, and \(y\) be an ordinary arc on \(S\). Then \(x_i\) can be written as

\[
x_i = \frac{1}{\text{cross}(T, y)} \sum_{P} x(P) y(P)
\]

where \(P\) is a perfect matching of \(T\) and \(y\), and \(\text{cross}(T, y)\) is the number of crossings of \(T\) and \(y\).

**Generalized Cluster Algebras**

Fix a semifield \(\mathbb{F}\) and let \(F = \mathbb{Q}(x_1, \ldots, x_n)\). A generalized cluster seed in \(F\) is a quadruple \((\gamma, x, y, \mathcal{Z})\) where \(x, y\), and \(\mathcal{Z}\) are defined as in ordinary cluster algebras and \(\mathcal{Z}\) is a collection of exchange polynomials

\[
\mathcal{Z}(u) = x_{i0} + x_i u + \cdots + x_{i\mathcal{Z}} u^\mathcal{Z}
\]

with all \(x_{ij} \in \mathbb{F}\) and \(x_{i0} = x_i = 1\).

Generalized cluster algebras with all \(d_i \in \{0, 1\}\) can be modeled as a triangulated orbifolds via the dictionary:

- initial generalized cluster seed \(\rightarrow\) initial triangulation
- other cluster variables \(\rightarrow\) other arcs on the orbifold
- mutation \(m \rightarrow\) “flipping” arc \(\tau_m\)

The cluster variable \(x_i\) corresponds to an ordinary arc if \(d_i = 1\) and to a pending arc if \(d_i = 2\).

**Orbifolds**

An orbifold is a generalization of a manifold where the local structure is given by quotients of open subsets of \(\mathbb{R}^n\) under finite group actions.

Each orbifold point, denoted as \(p\), has associated integer order \(p\). Intuitively, an orbifold point of order \(p\) is \(1/p\)-th of a point. A winding arc with \(k\) self-intersections “sees” the orbifold point as a puncture if \(k < p\) and as an ordinary point if \(k = p\).

**Snake Graphs from Orbifolds**

Snake graphs from triangulated orbifolds are built from the following puzzle pieces:

- \(x_i\) labels
- \(\tau_m\) labels
- \(\bar{\tau}\) labels
- \(\mathcal{Z}\) labels

**Examples of Snake Graphs**

A generalized arc on a triangulated orbifold with orbifold points of order 3 (green) and order 4 (red)

**Orbifold Frieze Patterns**

A frieze pattern is an array of infinite rows of numbers which satisfy a certain local arithmetic property. A particular frieze pattern is uniquely determined by its arithmetic property and its quiddity row, the first non-trivial row.

We can define the quiddity sequence associated to a triangulated orbifold by taking the concatenation of each two consecutive boundary components, computing their Laurent expansion, and then specializing all variables to one. This will uniquely determine a frieze pattern whose entries are specializations of elements of the generalized cluster algebra.

These frieze patterns are finite if and only if the orbifold surface has zero or one orbifold points and is homeomorphic to a disk with marked points on the boundary.

Infinite frieze patterns have a family of invariants called growth coefficients. As in the surface case, our band graphs encode the growth coefficients.

**Future Directions**

- Can this construction be extended to generalized cluster algebras that don’t correspond to triangulated orbifolds?
- Do our snake graphs help describe algebraic structure in a polygon-dissected surface?
- Is there an elementary way to classify the frieze patterns we can construct in this manner?

**References**