A q-deformed type B Cauchy identity and Chow's quasisymmetric functions

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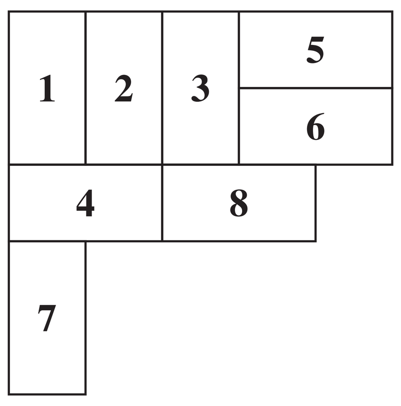
Type B quasisymmetric functions & domino tableaux

The **hyperoctahedral group** B_n is the set of signed permutations. The **descent set** of $\pi \in B_n$ is $Des(\pi) = \{0 \le i \le n-1 \mid \pi(i) > \pi(i+1)\}$ with $\pi(0) = 0$ and its **total color** is $tc(\pi) = |\{0 \le i \le n \mid \pi(i) < 0\}|$.
Define an alphabet of indeterminates $X = \{x_0, x_1, x_2, \ldots\}$.
The **Chow's type B fundamental quasisymmetric function** [1] indexed by the subset $I \subseteq \{0\} \bigcup [n-1]$ is given by

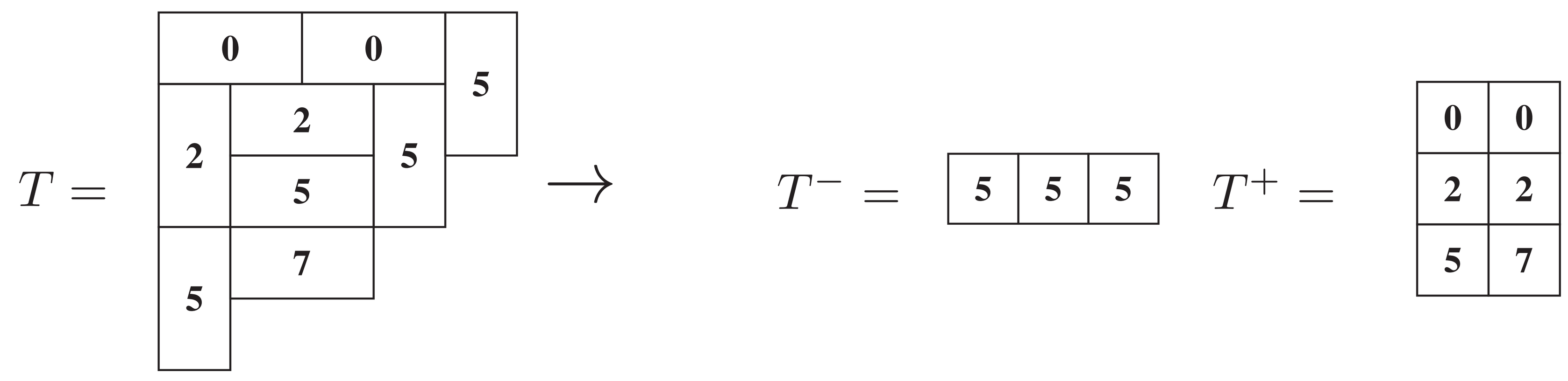
$$F_I^B(X) = \sum_{\substack{0 \le i_1 \le i_2 \le \cdots \le i_n \\ j \in I \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

A **standard domino tableau** is a Young diagram tiled by 2×1 (horizontal) and 1×2 (vertical) dominos filled with the elements of $[n]$ such that the entries are increasing along rows and columns.
The **descent set** of a standard domino tableau T is $Des(T) = \{1 \le i \le n-1 \mid i$ is in a strictly higher row than $i+1\} \cup \{0,$ if the domino '1' is vertical$\}$.
The **spin** $sp(T)$ of a domino tableau T is half the number of its vertical dominos.



The standard domino tableau on the LHS has a shape equal to $(5, 5, 4, 1, 1)$, a descent set equal to $\{0, 3, 5, 6\}$, and a spin equal to 2.

A **semistandard domino tableau** is a Young diagram tiled by dominos filled with integers greater or equal to 0. Vertical dominos may not be labelled with 0.
There is a well known bijection due to Stanton and White [2] between semistandard domino tableaux and pairs of semistandard Young tableaux:



The **number of negative dominos** $neg(T)$ of semistandard domino tableau T is the number of boxes in T^- and depends only on the shape of T.
The **Barbash and Vogan correspondence** [3] is a bijection between signed permutations $\pi \in B_n$ and pairs of standard domino tableaux P, Q such that

- $Des(\pi) = Des(P)$ and $Des(\pi^{-1}) = Des(Q)$ (descent preservation),
- $tc(\pi) = sp(P) + sp(Q)$ (color-to-spin property, [4]).

Domino functions (generating functions for semistandard domino tableaux)

$$\mathcal{G}_\lambda(X; q) = \sum_{T \in SSDT(\lambda)} q^{sp(T)} X^T = \sum_{T \in SDT(\lambda)} q^{sp(T)} F^B_{Des(T)}(X).$$

Our domino function ressemble the **LLT polynomials** introduced in [5] but **are not equal** because of the possible 0 labels in the semistandard domino tableaux. In particular our domino functions are **not symmetric** as the variable x_0 has a particular rôle.

References

[1] C-O. Chow, Non commutative symmetric functions of type B, 2001
[2] D. Stanton, D. White, A Schensted algorithm for rim-hook tableaux, 1985
[3] D. Barbash, D. Vogan, Primitive ideals and orbital integrals in complex classical groups, 1982
[4] M. Shimozono, D. White, A color-to-spin domino Schensted algorithm, 2001
[5] A. Lascoux, B. Leclerc, J.Y. Thibon, Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras and unipotent varieties, 1996
[6] T. Lam, On Sjostrand's skew sign-imbalance identity, 2006
[7] A. Mayorova, E. Vassilieva, A new link between the descent algebra of type B, domino tableaux and Chow's quasisymmetric functions, 2019

q-deformed Type B Cauchy identity

$X = \{\ldots, x_{-1}, x_0, x_1, \ldots\}$ is said **q-symmetric** if $x_{-i} = q x_i$ for $i > 0$.

Thm (M,V). *Let X and Y be two q-symmetric alphabets, we have:*

$$\mathcal{G}_{(2n)}(XY; q) = \sum_{\lambda \in \mathcal{P}^0(n)} \mathcal{G}_\lambda(X; q) \mathcal{G}_\lambda(Y; q).$$

Proof (sketch). Relate our domino functions (with 0 entries) and LLT polynomials (without the 0)

$$\mathcal{G}_\lambda(X; q) = \sum_{\substack{k \ge 0, \\ \lambda/2k \in \mathcal{P}^0(n-k)}} x_0^k LLT_{\lambda/2k}(X; q).$$

Then use skew domino Cauchy formula for LLT polynomials [6]

$$\sum_\lambda LLT_{\lambda/\alpha}(X; q) LLT_{\lambda/\beta}(Y; q) = \prod_{i,j>0} \frac{1}{(1 - x_i y_j)(1 - q x_i y_j)} \sum_\mu LLT_{\beta/\mu}(X; q) LLT_{\alpha/\mu}(Y; q).$$

Statistics for domino tableaux

We use the q-deformed type B Cauchy identity to prove the equidistribution of certain statstics on domino tableaux.

Cor (M,V). Decompose the two sides of the q-deformed type B Cauchy identity to get an analytic analogue to the result of Barbash and Vogan

$$\sum_{\pi \in B_n} q^{tc(\pi)} F^B_{Des(\pi)}(X) F^B_{Des(\pi^{-1})}(Y) = \sum_{\substack{\lambda \in \mathcal{P}^0(n) \\ T, U \in SDT(\lambda)}} q^{sp(T) + sp(U)} F^B_{Des(T)}(X) F^B_{Des(U)}(Y).$$

Proof (sketch). Show for X, Y q-symmetric

$$\mathcal{G}_{(2n)}(XY; q) = F^B_\emptyset(XY) = \sum_{\pi \in B_n} q^{tc(\pi)} F^B_{Des(\pi)}(X) F^B_{Des(\pi^{-1})}(Y).$$

Thm (M,V). *Let n be a positive integer. There is a one-to-one correspondence between couples of standard domino tableaux T, U of the same shape $\lambda \in \mathcal{P}^0(n)$ and couples of standard domino tableaux R, S of the same shape $\mu \in \mathcal{P}^0(n)$ (possibly $\lambda \neq \mu$) such that $Des(T) = Des(R)$, $Des(U) = Des(S)$ and $sp(T) + sp(U) = neg(R) = neg(S)$.*

Type B q-Schur positivity

Def. *A pair composed of a set $\mathcal{B} \subset B_n$ and a statistic stat defined on $\mathcal{B}$ is* **q-$\mathcal{G}$ positive** *if the function $Q(\mathcal{B}, stat)(X) = \sum_{\pi \in \mathcal{B}} q^{stat(\pi)} F^B_{Des(\pi)}(X)$ can be written as*

$$Q(\mathcal{B}, stat)(X) = \sum_\lambda c_\lambda(q) \mathcal{G}_\lambda(X; q),$$

where the $c_\lambda(q)$ are polynomials in $q^{\frac{1}{2}}$ with non-negative integer coefficients.

Thm (M,V). *Set $stat = tc$. Type B inverse descent classes and type B Knuth classes are q-$\mathcal{G}$ positive.*

Proof (sketch). For $J \subset \{0\} \cup [n-1]$ show

$$\sum_{\pi \in D^n_J} q^{tc(\pi)} F^B_{Des(\pi)}(X) = \sum_{Des(U) = J} q^{sp(U)} \mathcal{G}_{sh(U)}(X; q).$$

Thm (M,V). *For a permutation $\pi \in B_n$, let $l(\pi)$ be its type B Coxeter length. The pair (B_n, l) is type B q-Schur positive and*

$$\sum_{\pi \in B_n} q^{l(\pi)} F^B_{Des(\pi)}(X) = \sum_{\substack{\lambda \in \mathcal{P}^0(n), \\ Q \in SDT(\lambda)}} q^{2maj(Q) + sp(Q)} \mathcal{G}_{sh(Q)}(X; q).$$