A generalized quantum cohomology algebra for $\mathrm{Gr}(k,n)$

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Manifest

The purpose of this work is to generalize the algebraic/combinatorial parts of Schubert calculus: the properties of the cohomology ring $\mathrm{H}^*(\mathrm{Gr}(k,n))$ and the quantum cohomology ring $\mathrm{QH}^*(\mathrm{Gr}(k,n))$ of the Grassmannian.
Specifically, we find a new algebra $\mathcal{S}/I$ that deforms both of these rings and shares multiple properties with them (a Schur basis, a Pieri rule, a "rim hook algorithm", an S_3-symmetry).
We define and study $\mathcal{S}/I$ purely algebraically. Nevertheless, its structure constants appear to be positive, which suggests some yet unknown geometry lurking behind it.

The general setup

- Let $\mathbf{k}$ be a commutative ring. Let $\mathbb{N} = \{0,1,2,\ldots\}$. Let $n \geq k \geq 0$.
- Let $\mathcal{P} = \mathbf{k}[x_1, x_2, \ldots, x_k]$ be the polynomial ring in k indeterminates over $\mathbf{k}$.
- For each (finite or infinite) sequence α and each i, let α_i be the i-th entry of α.
- For each $\alpha \in \mathbb{N}^k$, let x^α be the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$.
- For each $\alpha \in \mathbb{Z}^k$, let $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$.
- Let $\mathcal{S}$ denote the ring of *symmetric* polynomials in $\mathcal{P}$.
- The commutative $\mathbf{k}$-algebra $\mathcal{S}$ is freely generated by the complete homogeneous symmetric polynomials $h_1, h_2, \ldots, h_k$.
- Let $a_1, a_2, \ldots, a_k \in \mathcal{P}$ such that $\deg a_i < n - k + i$ for all i. (For example, this holds if $a_i \in \mathbf{k}$.)
- Let J be the ideal of $\mathcal{P}$ generated by the k differences

$$h_{n-k+1} - a_1, \quad h_{n-k+2} - a_2, \quad \ldots, \quad h_n - a_k.$$

Theorem 1 (monomial basis of $\mathcal{P}/J$)

The $\mathbf{k}$-module $\mathcal{P}/J$ is free with basis $\left(\overline{x^\alpha}\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$, where the overline $\overline{}$ means "projection" onto whatever quotient we need (here: from $\mathcal{P}$ onto $\mathcal{P}/J$).

Specializing to $a_i \in \mathcal{S}$

- **FROM NOW ON, assume that** $a_1, a_2, \ldots, a_k \in \mathcal{S}$.
- Let I be the ideal of $\mathcal{S}$ generated by the k differences

$$h_{n-k+1} - a_1, \quad h_{n-k+2} - a_2, \quad \ldots, \quad h_n - a_k.$$

- A *k-partition* means a weakly decreasing k-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$. (Equivalently, it means a partition of length $\leq k$.)
- Let $\omega = \underbrace{(n-k, n-k, \ldots, n-k)}_{k \text{ entries}}$ and

$$P_{k,n} = \{\lambda \text{ is a } k\text{-partition} \mid \lambda_1 \leq n-k\} = \{k\text{-partitions } \lambda \subseteq \omega\}.$$

- For any k-tuple ν of integers (not necessarily a partition), set

$$h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_k} \qquad \text{and} \qquad s_\nu = \det\left(\left(h_{\nu_i - i + j}\right)_{1 \leq i \leq k,\ 1 \leq j \leq k}\right).$$

  Note that each h_ν is either 0 or can be rewritten as h_λ for some k-partition λ. Similarly for s_ν (except it can now be $\pm s_\lambda$ or 0).
- The $\mathbf{k}$-module $\mathcal{S}$ has a basis $(s_\lambda)_{\lambda \text{ is a } k\text{-partition}}$ of Schur polynomials.

Theorem 2 (Schur basis of $\mathcal{S}/I$)

The $\mathbf{k}$-module $\mathcal{S}/I$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$. (It will be called the *Schur basis*.)

Specializing to $a_i \in \mathbf{k}$

- **FROM NOW ON, assume that** $a_1, a_2, \ldots, a_k \in \mathbf{k}$.

This setting still is general enough to encompass ...

- **classical cohomology:** If $\mathbf{k} = \mathbb{Z}$ and $a_1 = a_2 = \cdots = a_k = 0$, then $\mathcal{S}/I \cong \mathrm{H}^*(\mathrm{Gr}(k,n))$; the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ corresponds to the Schubert classes.
- **quantum cohomology:** If $\mathbf{k} = \mathbb{Z}[q]$ and $a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k = -(-1)^k q$, then $\mathcal{S}/I \cong \mathrm{QH}^*(\mathrm{Gr}(k,n))$.

The above theorem lets us work in these rings without relying on geometry.

Structure constantts

- For every k-partition $\nu = (\nu_1, \nu_2, \ldots, \nu_k) \in P_{k,n}$, we define its *complement*

$$\nu^\vee := (n-k-\nu_k, n-k-\nu_{k-1}, \ldots, n-k-\nu_1) \in P_{k,n}.$$

- We let $g_{\alpha,\beta,\gamma} \in \mathbf{k}$ be the structure constants of $\mathcal{S}/I$ with respect to the Schur basis, labelled in such a way that

$$\overline{s_\alpha s_\beta} = \sum_{\gamma \in P_{k,n}} g_{\alpha,\beta,\gamma} \overline{s_{\gamma^\vee}} \qquad \text{for all } \alpha, \beta \in P_{k,n}.$$

  These generalize the Littlewood–Richardsons and (3-point) Gromov–Wittens.

Theorem 3 (S_3-symmetry of the structure constants)

For any $\alpha, \beta, \gamma \in P_{k,n}$, we have

$$g_{\alpha,\beta,\gamma} = g_{\alpha,\gamma,\beta} = g_{\beta,\alpha,\gamma} = g_{\beta,\gamma,\alpha} = g_{\gamma,\alpha,\beta} = g_{\gamma,\beta,\alpha} = \operatorname{coeff}_\omega\left(\overline{s_\alpha s_\beta s_\gamma}\right).$$

Here, $\operatorname{coeff}_\omega f$ means the $\overline{s_\omega}$-coefficient when $f \in \mathcal{S}/I$ is expanded in the Schur basis.

Theorem 4 (h-basis of $\mathcal{S}/I$)

The $\mathbf{k}$-module $\mathcal{S}/I$ is free with basis $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$.

Proposition (h-reduction)

Let m be a positive integer. Then,

$$\overline{h_{n+m}} = \sum_{j=0}^{k-1} (-1)^j a_{k-j} \overline{s_{(m,1^j)}},$$

where $(m, 1^j) := (m, \underbrace{1, 1, \ldots, 1}_{j \text{ ones}}, 0, 0, 0, \ldots)$ (a hook-shaped k-partition).

Theorem 5 (h-Pieri rule, generalizing [1, (22)])

Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\nu \subseteq \lambda} c^\lambda_{(n-k-j+1, 1^{i-1}), \nu} \overline{s_\nu},$$

where $c^\gamma_{\alpha,\beta}$ are the usual Littlewood–Richardson coefficients.

Theorem 6 (rim hook rule, generalizing [1, §2])

Let μ be a k-partition with $\mu_1 > n-k$. Let

$$W = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k \mid \lambda_1 = \mu_1 - n \text{ and } \lambda_i - \mu_i \in \{0,1\} \text{ for all } i \in \{2, 3, \ldots, k\}\}.$$

(Not all $\lambda \in W$ are k-partitions, but all belong to $\mathbb{Z}^k$, so s_λ makes sense.) Then,

$$\overline{s_\mu} = \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\lambda \in W; \\ |\lambda| = |\mu| - (n-k+j)}} \overline{s_\lambda}.$$

Conjecture 7 (positivity? confirmed for $n \leq 8$)

Let $b_i = (-1)^{n-k-1} a_i$ for each $i \in \{1, 2, \ldots, k\}$. Let $\lambda, \mu, \nu \in P_{k,n}$. Is $(-1)^{|\lambda|+|\mu|-|\nu|} \operatorname{coeff}_\nu\left(\overline{s_\lambda s_\mu}\right)$ a polynomial in $b_1, b_2, \ldots, b_k$ with coefficients in $\mathbb{N}$?

More conjectures and results in [2].

[1] Aaron Bertram, Ionut Ciocan-Fontanine, William Fulton, *Quantum multiplication of Schur polynomials*, Journal of Algebra **219** (1999), pp. 728–746.

[2] Darij Grinberg, *A basis for a quotient of symmetric polynomials (draft)*, `http://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf`

[3] Alexander Postnikov, *Affine approach to quantum Schubert calculus*, Duke Mathematical Journal **128**, No. 3, pp. 473–509.

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