Plabic R-Matrices

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Motivation

One of the key features of the theory of plabic networks developed by Postnikov is the fact that if a plabic graph is reduced, the face weights can be uniquely recovered from boundary measurements. On surfaces more complicated than a disk this property is lost. We study a certain semi-local transformation of weights for plabic networks on a cylinder that preserve boundary measurements. We call this a *plabic R-matrix*. Plabic R-matrices have underlying cluster algebra structure, generalizing recent work of Inoue-Lam-Pylyavskyy. Special cases of transformations we consider include geometric R-matrices appearing in Berenstein-Kazhdan theory of geometric crystals, and also certain transformations appearing in a recent work of Goncharov-Shen.

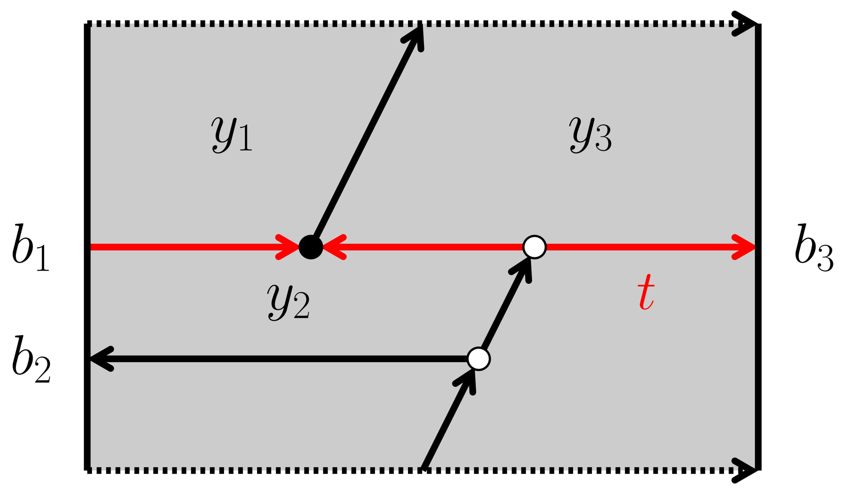
Directed Plabic Networks on a Cylinder

Definition

A *directed plabic network on a cylinder* is a directed graph that can be embedded in the cylinder. Every vertex is colored black or white such that black vertices have exactly one outgoing arrow and white vertices have exactly one incoming arrow. The faces have positive real face weights $\{y_f\}$ that multiply to 1 and there is a trail with weight t.

Given a planar directed network on a cylinder with boundary vertices $b_1, \dots, b_n$, we can define the boundary measurements

$$M_{ij} = \sum_{\substack{\text{paths } P \text{ from} \\ b_i \text{ to } b_j}} (-1)^{wind(C_P)-1} \zeta^{int(P)} wt(P, y, t).$$

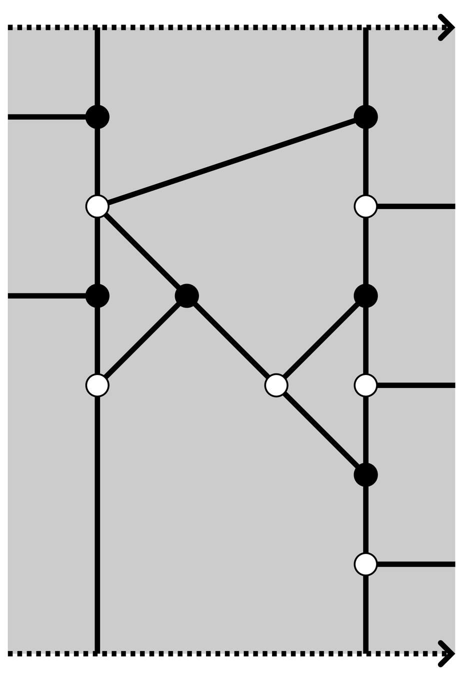


$$M_{12} = \frac{y_2\zeta}{y_1 y_2 + \zeta}$$

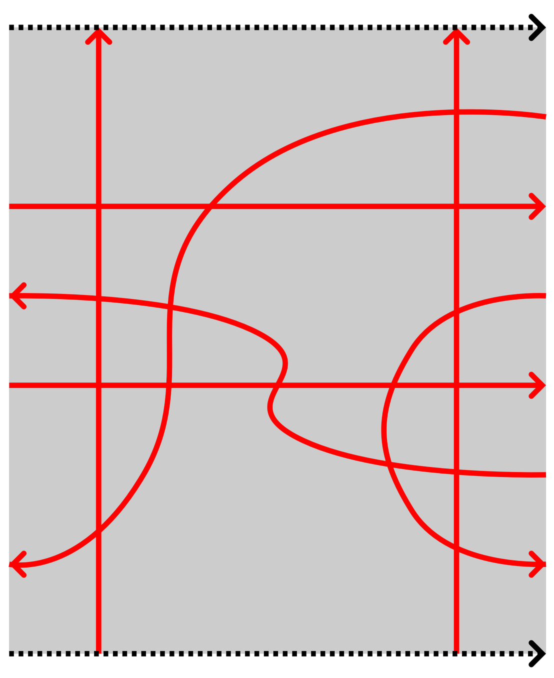
$$M_{13} = \frac{-t y_3 \zeta}{1 + y_3 \zeta}$$

Inverse Boundary Problem: What information about a planar directed network can be recovered given the boundary measurements?

Cylindric k-loop Plabic Graphs



plabic graph

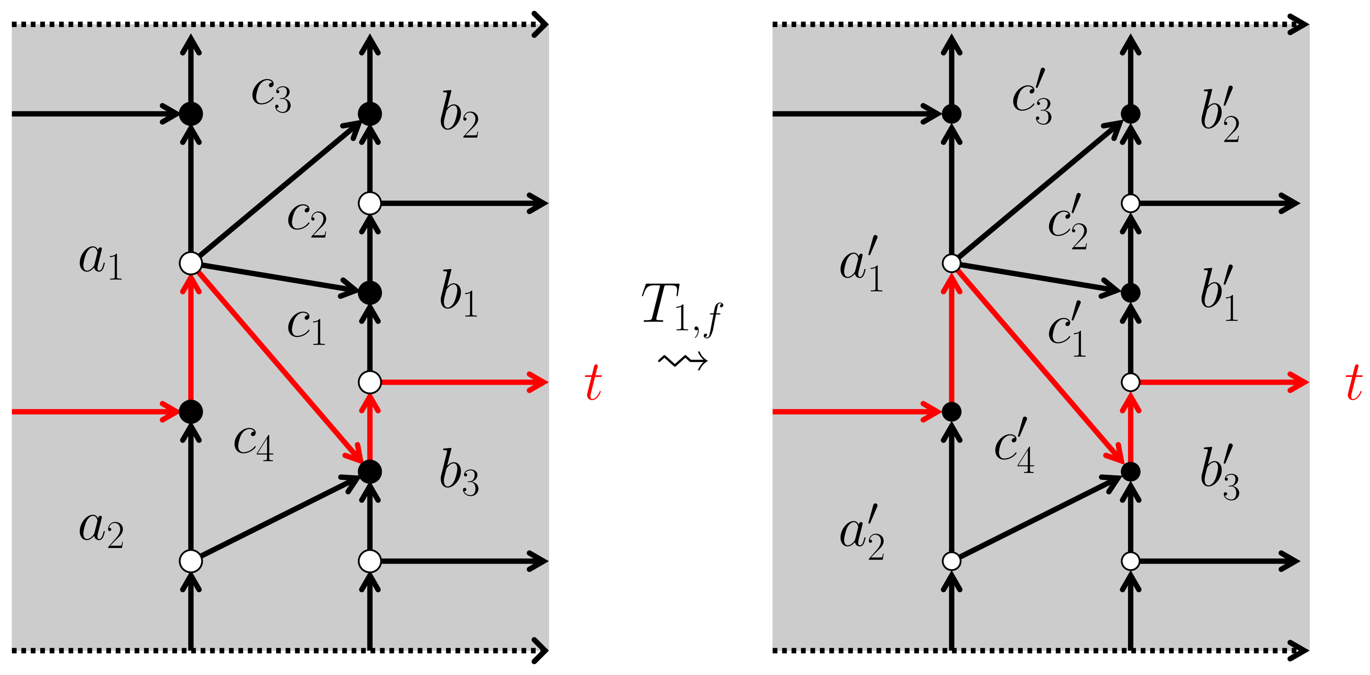


Postnikov diagram

Theorem

Any cylindric k-loop plabic graph can be transformed by moves into one that has no interior vertices.

Plabic R-Matrices



$$a_1' = \frac{a_1a_2b_1b_2b_3c_1c_2c_3 + 1 + c_1 + c_1c_2}{a_2b_1b_2b_3(1 + c_1 + c_1c_2 + c_1c_2c_3)}$$

$$b_1' = \frac{b_1(a_1a_2b_1b_2b_3c_1c_2 + a_1a_2b_1b_2b_3c_1c_2c_3 + 1 + c_1)}{a_1a_2b_1b_2b_3c_1 + a_1a_2b_1b_2b_3c_1c_2 + a_1a_2b_1b_2b_3c_1c_2c_3 + 1}$$

$$c_1' = \frac{a_1a_2b_1b_2b_3c_1(1 + c_1 + c_1c_2 + c_1c_2c_3)}{a_1a_2b_1b_2b_3c_1c_2 + a_1a_2b_1b_2b_3c_1c_2c_3 + 1 + c_1}$$

Theorem

Plabic R-matrices have the following properties:

1. They preserve the boundary measurements.
2. They are involutions.
3. (a, b, c, t) and (a', b', c', t) are the only choices of face and trail weights on a fixed cylindric 2-loop plabic network with the canonical orientation that preserve the boundary measurements.
4. They satisfy the braid relation. That is, $T_{\varkappa,f}T_{\varkappa+1,f}T_{\varkappa,f} = T_{\varkappa+1,f}T_{\varkappa,f}T_{\varkappa+1,f}$ for $1 \le \varkappa < k-1$.

Cluster Structure

Plabic R-matrices are deeply connected to the y-dynamics of certain cluster algebras. Let Q be the dual quiver to a cylindric k-loop plabic network with no interior vertices. If $1 \le \varkappa < k$, label the vertices corresponding to the faces between strings $\varkappa$ and $\varkappa+1$ as $1, \dots, n$ around the cylinder. We define the transformation

$$\tau_\varkappa := \mu_1\mu_2 \dots \mu_{n-2} s_{n-1,n} \mu_n \mu_{n-1} \dots \mu_1.$$

Theorem

Let Q be the dual quiver to a cylindric k-loop plabic network with no interior vertices. If we set the y-variable for each vertex equal to the weight of the corresponding face and apply $\tau_\varkappa$, the y-variables we obtain are the same as the face variables with the transformation $T_{\varkappa,f}$ applied to them.

References

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