ON RANDOM SHIFTED STANDARD YOUNG TABLEAUX AND 132-AVOIDING SORTING NETWORKS

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The limit shape of shifted staircase SYT

- The *shifted Young diagram* λ^{sh} of a strictly decreasing partition λ is obtained from the Young diagram by shifting rows to the right, row i by $i-1$ steps.
- A shifted standard Young tableau (SYT) of shape λ^{sh} is an increasing filling of the shifted diagram λ^{sh} with $1, 2, \ldots, |\lambda|$.
- $\Delta_n = (n-1, \ldots, 2, 1)$ is the *staircase* partition.

1	2	4	5
	3	6	7
		8	9
			10

Figure 1: A SYT of the shifted staircase shape Δ_5^{sh}.

One may view the entries in a SYT as heights, defining a surface. The theorem below concerns the limit surface L of uniformly random shifted staircase SYT. We omit the somewhat technical definition of L. However, note that L is half the limit surface of random square SYT obtained by Pittel and Romik [5]. Our proof relies heavily on their work. For a SYT T, $T(i,j)$ denotes the entry in row i, column j.

Theorem 1

For $n \in \mathbb{N}$, let $N = \binom{n}{2}$, $\mathcal{T}_n$ denote the set of shifted SYT of shape Δ_n^{sh}, and $\mathbb{P}_n$ the uniform probability measure on $\mathcal{T}_n$. Then for all $\epsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}_n\left(T \in \mathcal{T}_n : \max_{(i,j)\in\Delta_n^{sh}} \left|\frac{T(i,j)}{N} - L\left(\frac{i}{n}, \frac{j}{n}\right)\right| > \epsilon\right) = 0\,. \qquad (1)$$

Moreover for all $p \in (0, 1/2)$ and all $q \in (0, p/2)$ such that $2p + q < 1$

$$\lim_{n\to\infty} \mathbb{P}_n\left(T \in \mathcal{T}_n : \max_{\substack{(i,j)\in\Delta_n^{sh} \\ \sigma(i/n, j/n) > n^{-q}}} \left|\frac{T(i,j)}{N} - L\left(\frac{i}{n}, \frac{j}{n}\right)\right| > n^{-p}\right) = 0\,, \qquad (2)$$

where $\sigma(x,y) = \min\{xy, (1-x)(1-y)\}$.

Equation (1) provides point-wise convergence to the limit surface, while (2) specifies the rate of convergence, assuming a sufficient distance to the sides.

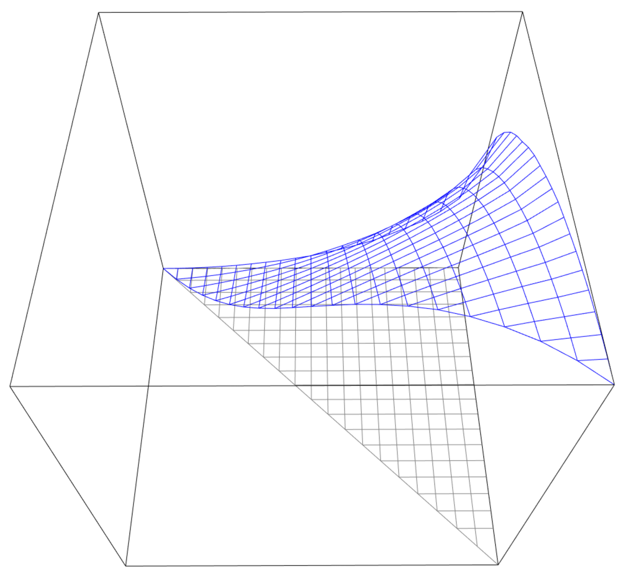

Figure 2: The limit shape of uniformly random shifted SYT of staircase shape.

132-avoiding sorting networks

- n-element *sorting networks* $w = w_1 \ldots w_N$ are reduced words of the reverse permutation $n\; n-1\; \ldots\; 2\; 1$.
- They are in bijection with SYT of shape Δ_n (e.g. [3]).
- If the *intermediate permutations* $\sigma_k = s_{w_1} \ldots s_{w_k}$, $1 \le k \le N$, of w are 132-avoiding, we say that w is an n-element *132-avoiding sorting network*. Denote the set of them by $\mathcal{R}_n^{132}$.
- $\mathcal{R}_n^{132}$ is in bijection with shifted SYT of shape Δ_n^{sh} [4]. For example,

$w = 1213423121 \longrightarrow$

1	2	4	5
3	6	7	
8	9		
10			

$\longrightarrow$

1	2	4	5
	3	6	7
		8	9
			10

  Note that e.g. the intermediate permutation $\sigma_4 = s_1 s_2 s_1 s_4 = 32154$ is 132-avoiding (and so are the other intermediate permutations of w).
- Random sorting networks were first studied by Angel, Holroyd, Romik and Virág [1]. Dauvergne proved their conjectures in [2]. We study similar questions on random 132-avoiding sorting networks.

Intermediate permutations and trajectories

The *diagram* $D(\sigma)$ of a permutation σ is the set of cells left unshaded when we shade the cells weakly to the east and south of 1-entries in the permutation matrix $M(\sigma)$. See e.g. Figure 4. We show that the limits of diagrams of intermediate permutations of random 132-avoiding sorting networks are given by shifting the level curves of the limit surface L as in Figure 3.

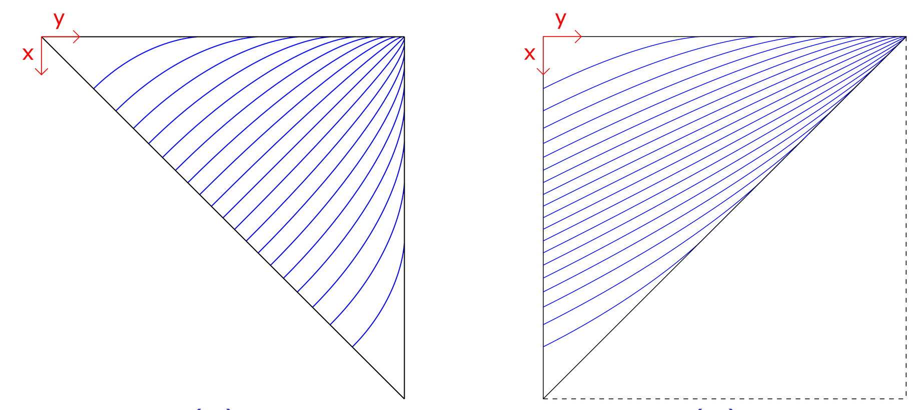

Figure 3: Translating the limit shape of shifted staircase SYT into the limits of diagrams of intermediate permutations. (a) Level curves $L(x,y) = \alpha$ of the limit surface L at $\alpha = 0.05, 0.1, \ldots, 0.95$. (b) The blue curves are the limit curves of the diagrams of the intermediate permutations $\sigma_{\lfloor \alpha N \rfloor}$ at times $\alpha = 0.05, 0.1, \ldots, 0.95$.

The limits of intermediate permutation matrices can then be obtained using the symmetry of the probability measure under reversing the sorting network.

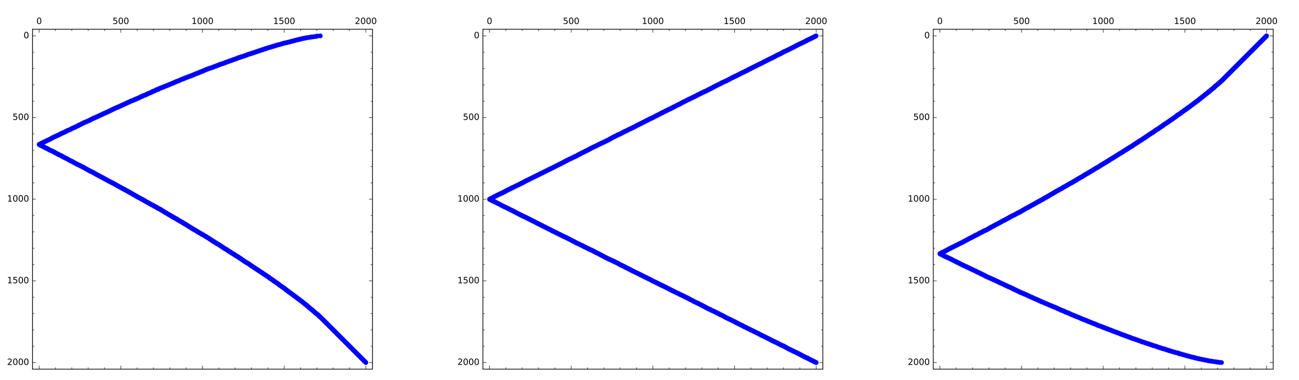

Figure 4: Intermediate permutation matrices of a random 132-avoiding sorting network with 1000 elements at times $\alpha = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$. The blue dots are the 1s, and the diagrams are the top-left white regions above the blue curves.

The *scaled trajectory* $f_i(\alpha)$ of i in $w \in \mathcal{R}_n^{132}$ is defined by $f_i(\alpha) = \sigma_{\alpha N}^{-1}(i)/n$ for $\alpha N \in \mathbb{Z}$, and by linear interpolation for other $\alpha \in [0,1]$. See the figures below. We prove that $f_1(\alpha)$ converges in probability to $2\sqrt{\alpha - \alpha^2}$ for $0 \le \alpha \le \frac{1}{2}$, and 1 for $\frac{1}{2} \le \alpha \le 1$.

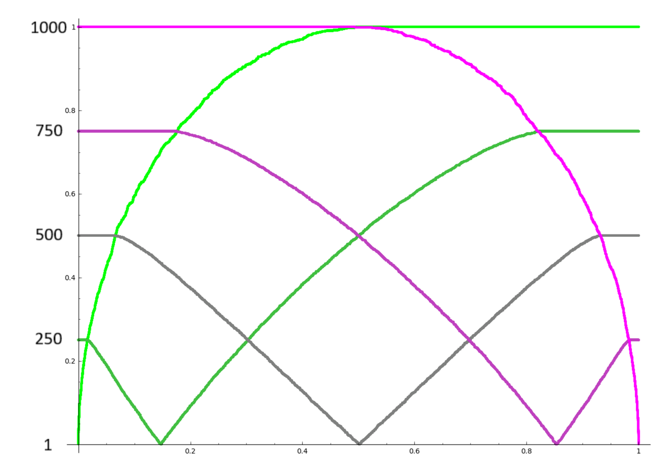

Figure 5: Scaled trajectories in a 132-avoiding sorting network with 1000 elements.

In fact, each trajectory can be traced as a certain intersection on the limit shape L, but given its technical definition, it is hard to compute arbitrary trajectories.

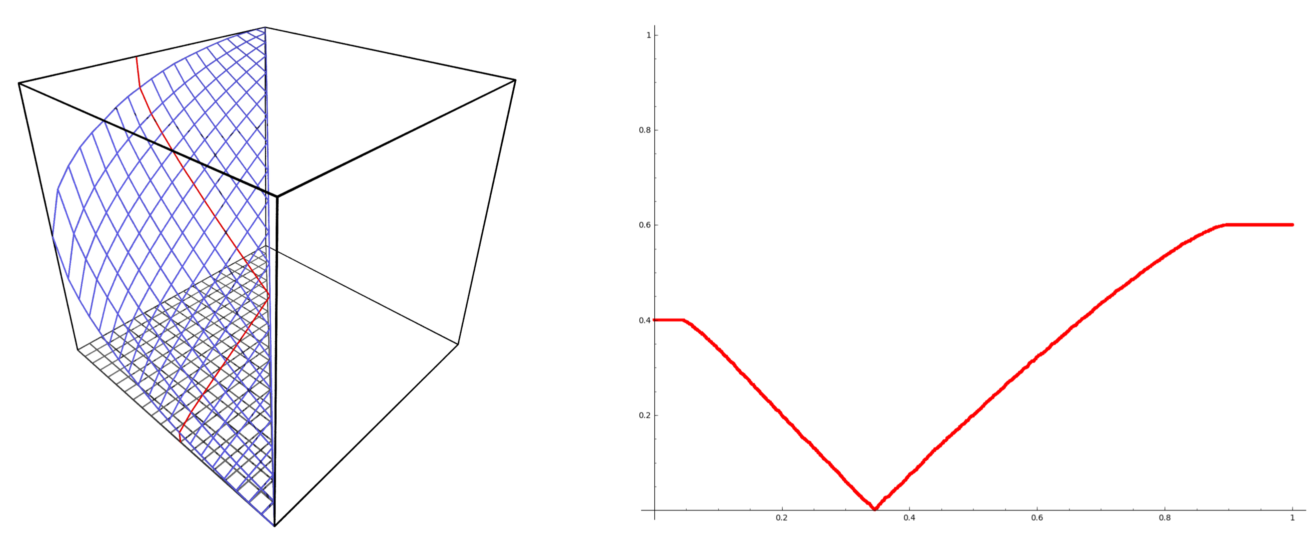

Figure 6: A curve (in red) determining the scaled trajectory shown on the right from the limit shape.

Adjacencies

- For $w \in \mathcal{R}_n^{123}$, $k \in [N-1]$ is called an *adjacency* if $|w_{k+1} - w_k| = 1$.
- Adjacencies in 132-avoiding sorting networks correspond directly to adjacencies in shifted staircase SYT.
- Let T be a (possibly shifted) SYT and (i,j) a cell in it. Then $(T,(i,j))$ is an *adjacency* if $T(i,j+1) = T(i,j)+1$ or $T(i+1,j) = T(i,j)+1$. For example, $(T,(1,2))$ and $(T,(1,3))$ are adjacencies in the SYT T in Figure 1.

Theorem 2

The expected number of horizontal (vertical) adjacencies in column $c < n-1$ (row $r < n-1$) of a uniformly random shifted staircase SYT is equal to 1.

Theorem 3

The expected number of adjacencies in a random 132-avoiding sorting network of length N is $2(n-2)$.

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