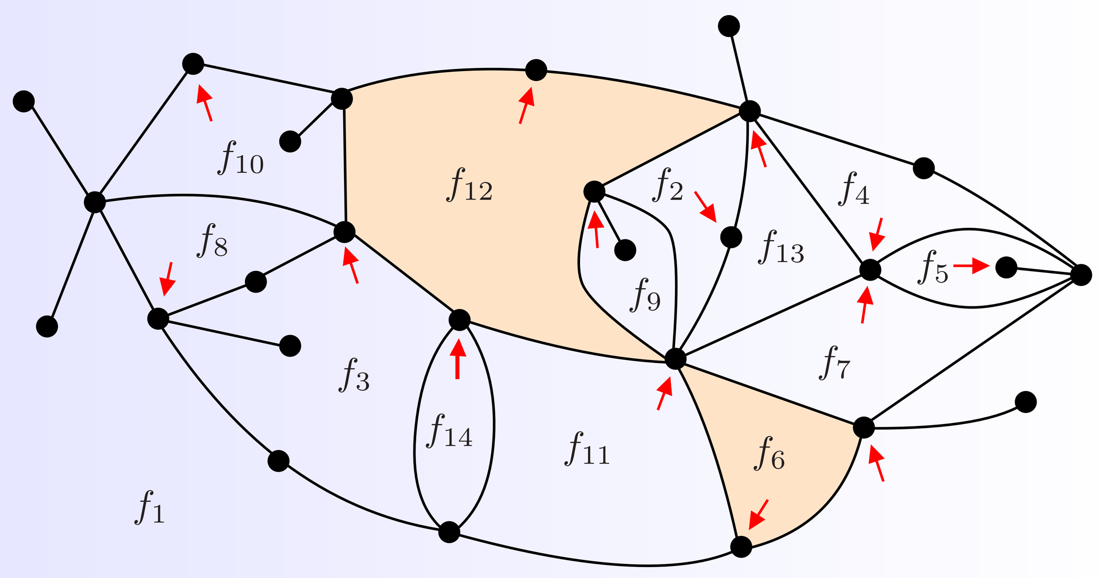
Slit-slide-sew bijections for bipartite and quasibipartite plane maps

Jérémie Bettinelli

Bipartite/quasibipartite plane map

Map of type $\boldsymbol{a} = (a_1, \dots, a_r)$: plane map with r numbered faces $f_1, \dots, f_r$ of respective degrees $a_1, \dots, a_r$, with a marked corner per face.

Bipartite: every a_i even. Quasibipartite: exactly two odd a_i's.



Aim

Bijectively interpret relations between the numbers of maps having almost the same type.

Tutte's formula of slicings [Tutte '62]

For $\boldsymbol{a} = (a_1, \dots, a_r) \in \mathbb{N}^r$, define the following.

- $M(\boldsymbol{a})$: number of maps of type $\boldsymbol{a}$.
- $E(\boldsymbol{a}) := \frac{1}{2}\sum_{i=1}^r a_i$: numbers of edges of maps of type $\boldsymbol{a}$.
- $V(\boldsymbol{a}) := E(\boldsymbol{a}) - r + 2$: numbers of vertices of maps of type $\boldsymbol{a}$.

Thm. *For bipartite or quasibipartite maps (i.e., at most two odd a_i's),*

$$M(\boldsymbol{a}) = \frac{(E(\boldsymbol{a}) - 1)!}{V(\boldsymbol{a})!} \prod_{i=1}^{r} \alpha(a_i)\,,$$

where $\alpha(x) := \dfrac{x!}{\lfloor x/2 \rfloor!\, \lfloor (x-1)/2 \rfloor!}$.

recovered by transfer bijections [Cori '75], encoding by blossoming trees [Schaeffer '97], Bouttier–Di Francesco–Guitter bijection [Collet–Fusy '14]

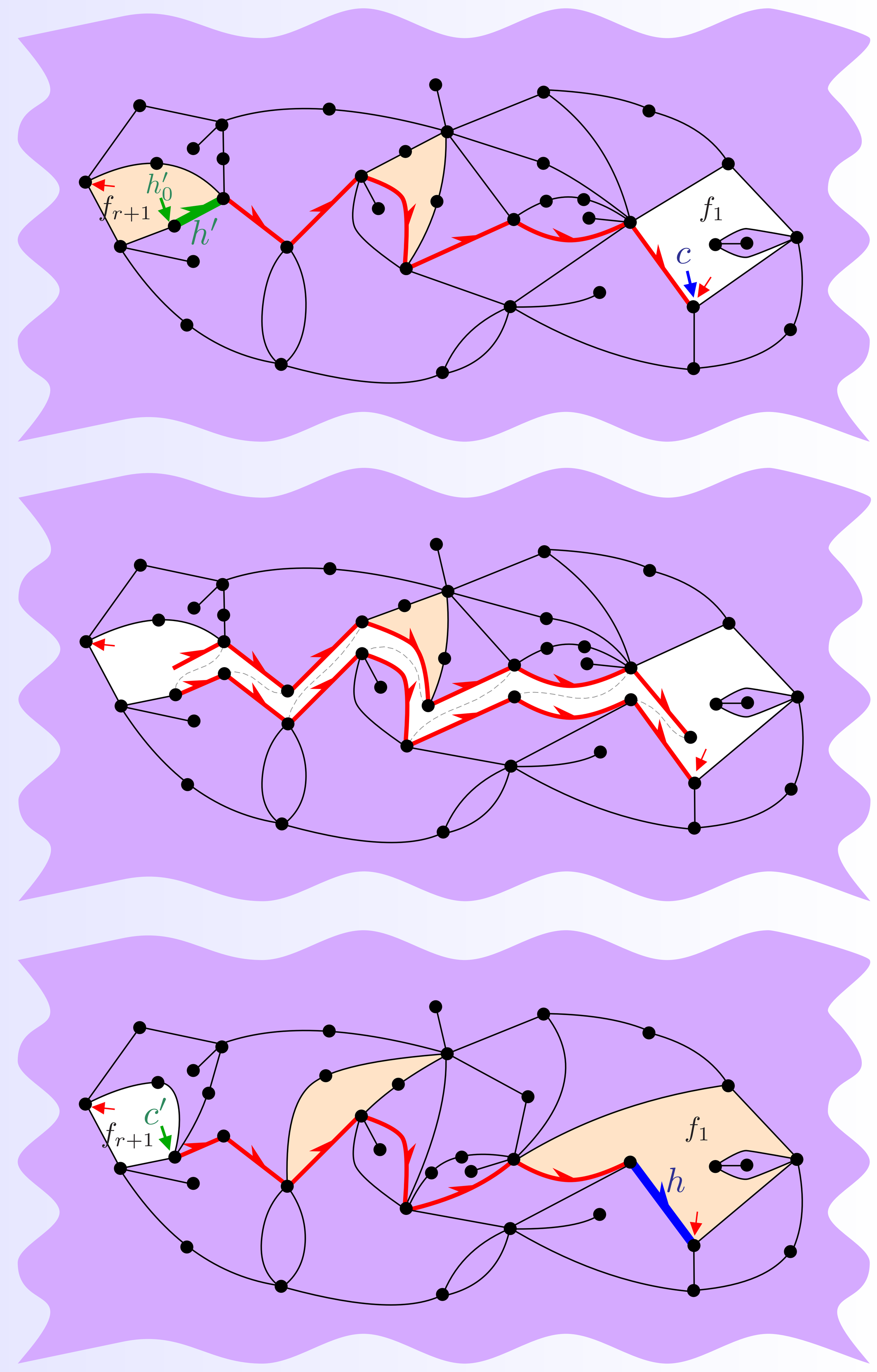
How to transfer a corner: degree ≥ 2

$\boldsymbol{a} = (a_1, \dots, a_{r+1}) \in \mathbb{N}^{r+1}$ with $a_{r+1} \geq 2$ such that

- either every a_i is even;
- or only a_{r+1} and one other a_i are odd.

$$\tilde{\boldsymbol{a}} = (\tilde{a}_1, \dots, \tilde{a}_r) := (a_1{+1}, a_2, \dots, a_r, a_{r+1}{-1})$$

$$\underbrace{(a_1+1)}_{\substack{\text{corner}\\ c \text{ in } f_1}}\ \underbrace{\lfloor a_{r+1}/2 \rfloor}_{\substack{\text{half-edge } h' \\ \text{of } f_{r+1} \text{ toward } c}}\ M(\boldsymbol{a}) = \underbrace{\lfloor \tilde{a}_1/2 \rfloor}_{\substack{\text{half-edge } h \\ \text{of } f_1 \text{ away from } c'}}\ \underbrace{(\tilde{a}_{r+1}+1)}_{\substack{\text{corner}\\ c' \text{ in } f_{r+1}}} M(\tilde{\boldsymbol{a}})$$

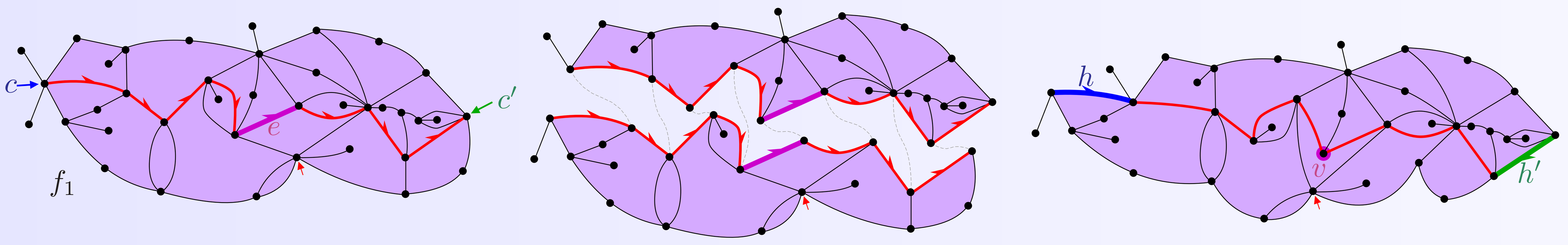


How to add two corners in a bipartite map

$\boldsymbol{a} = (a_1, \dots, a_r) \in 2\mathbb{N}^r$

$\tilde{\boldsymbol{a}} = (\tilde{a}_1, \dots, \tilde{a}_r) := (a_1{+2}, a_2, \dots, a_r)$

$$\underbrace{(a_1+1)}_{\substack{\text{corner}\\ c \text{ in } f_1}}\ \underbrace{(a_1+2)}_{\substack{\text{oth. corner}\\ c' \text{ in } f_1}}\ \underbrace{E(\boldsymbol{a})}_{\text{edge}}\ M(\boldsymbol{a}) = \underbrace{\lfloor \tilde{a}_1/2 \rfloor}_{\substack{\text{half-edge } h \\ \text{of } f_1 \text{ toward } v}}\ \underbrace{\lfloor (\tilde{a}_1-1)/2 \rfloor}_{\substack{\text{oth. half-edge } h' \\ \text{of } f_1 \text{ toward } v}}\ \underbrace{V(\tilde{\boldsymbol{a}})}_{\text{vertex } v}\ M(\tilde{\boldsymbol{a}})$$



(1.) Orient e toward c' and consider the rightmost geodesic from it toward c'.

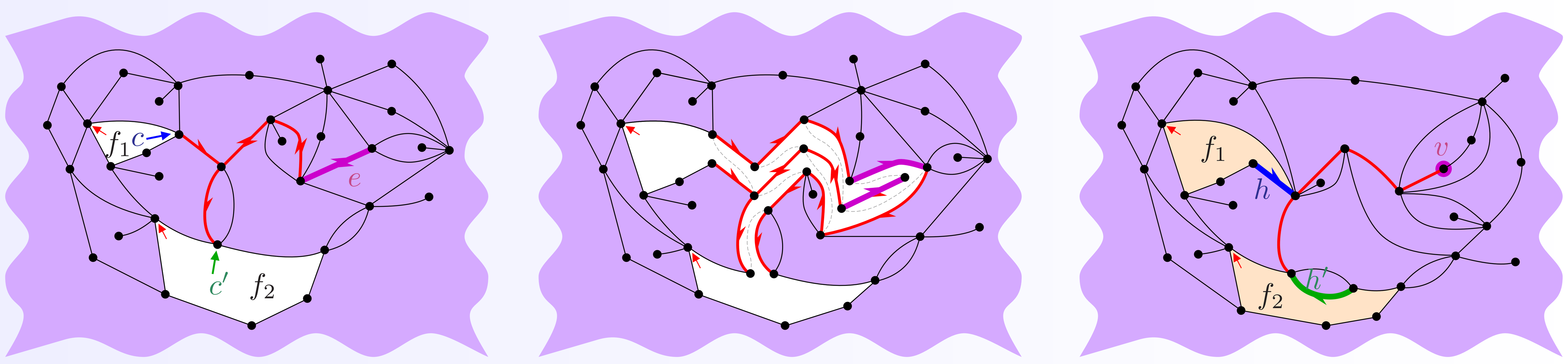
(2.) Do "the same" with c instead of c'.

(3.) Slit! Slide! Sew! And mark v, h and h'.

$\boldsymbol{a} = (a_1, \dots, a_r) \in 2\mathbb{N}^r$

$\tilde{\boldsymbol{a}} := (a_1{+1}, a_2{+1}, a_3, \dots, a_r)$

$$\underbrace{(a_1+1)}_{\substack{\text{corner}\\ c \text{ in } f_1}}\ \underbrace{(a_2+1)}_{\substack{\text{corner}\\ c' \text{ in } f_2}}\ \underbrace{E(\boldsymbol{a})}_{\text{edge}}\ M(\boldsymbol{a}) = \underbrace{\lfloor \tilde{a}_1/2 \rfloor}_{\substack{\text{half-edge } h \\ \text{of } f_1 \text{ toward } v}}\ \underbrace{\lfloor \tilde{a}_2/2 \rfloor}_{\substack{\text{half-edge } h' \\ \text{of } f_2 \text{ toward } v}}\ \underbrace{V(\tilde{\boldsymbol{a}})}_{\text{vertex } v}\ M(\tilde{\boldsymbol{a}})$$

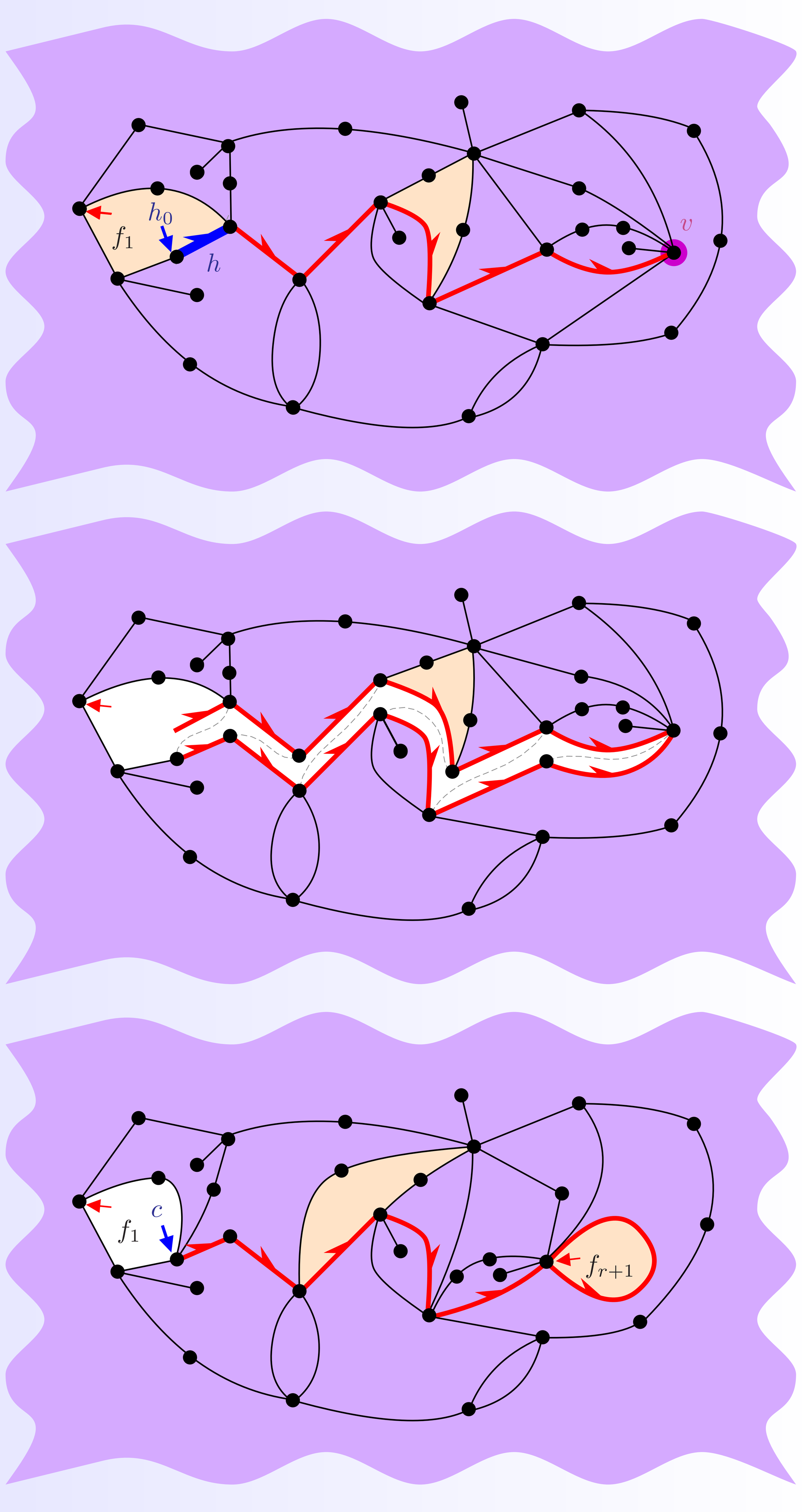


How to transfer a corner: degree 1

$\boldsymbol{a} = (a_1, \dots, a_r, 1) \in \mathbb{N}^{r+1}$ with two odd coordinates

$\tilde{\boldsymbol{a}} := (a_1{+1}, a_2, \dots, a_r)$

$$\underbrace{(a_1+1)}_{\substack{\text{corner}\\ c \text{ in } f_1}}\ M(\boldsymbol{a}) = \underbrace{\lfloor \tilde{a}_1/2 \rfloor}_{\substack{\text{half-edge } h \\ \text{of } f_1 \text{ toward } c'}}\ \underbrace{V(\tilde{\boldsymbol{a}})}_{\text{vertex } v}\ M(\tilde{\boldsymbol{a}})$$



Slit, slide, sew along the leftmost geodesic from h_0 to v.

Proof for transfer bijections (degree ≥ 2)

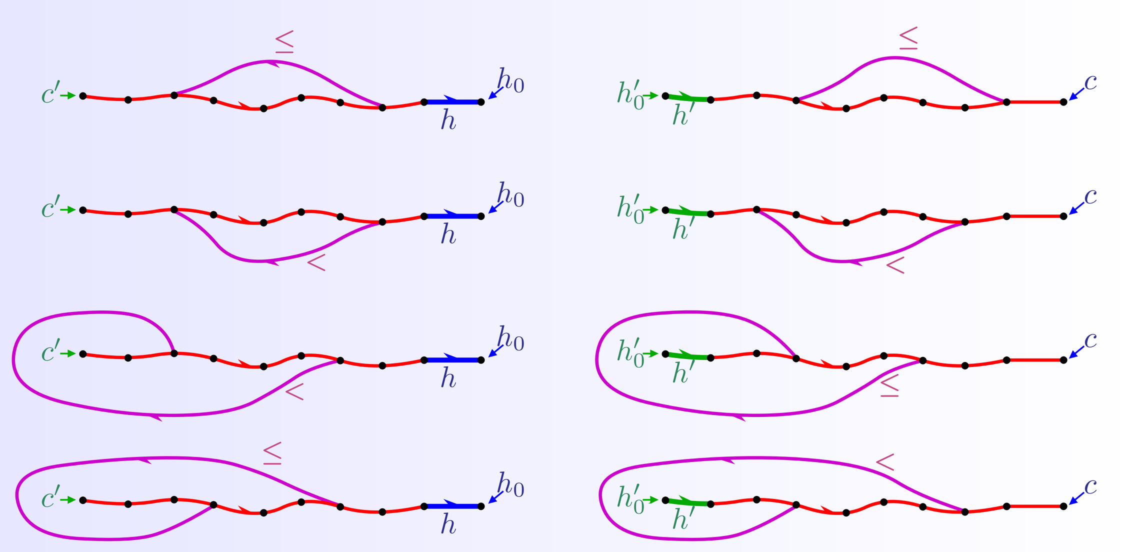
We need to see that the leftmost geodesic ℓ from h'_0 toward c becomes the rightmost geodesic r from h_0 toward c'.

Suppose by contradiction that r is not the image of ℓ.

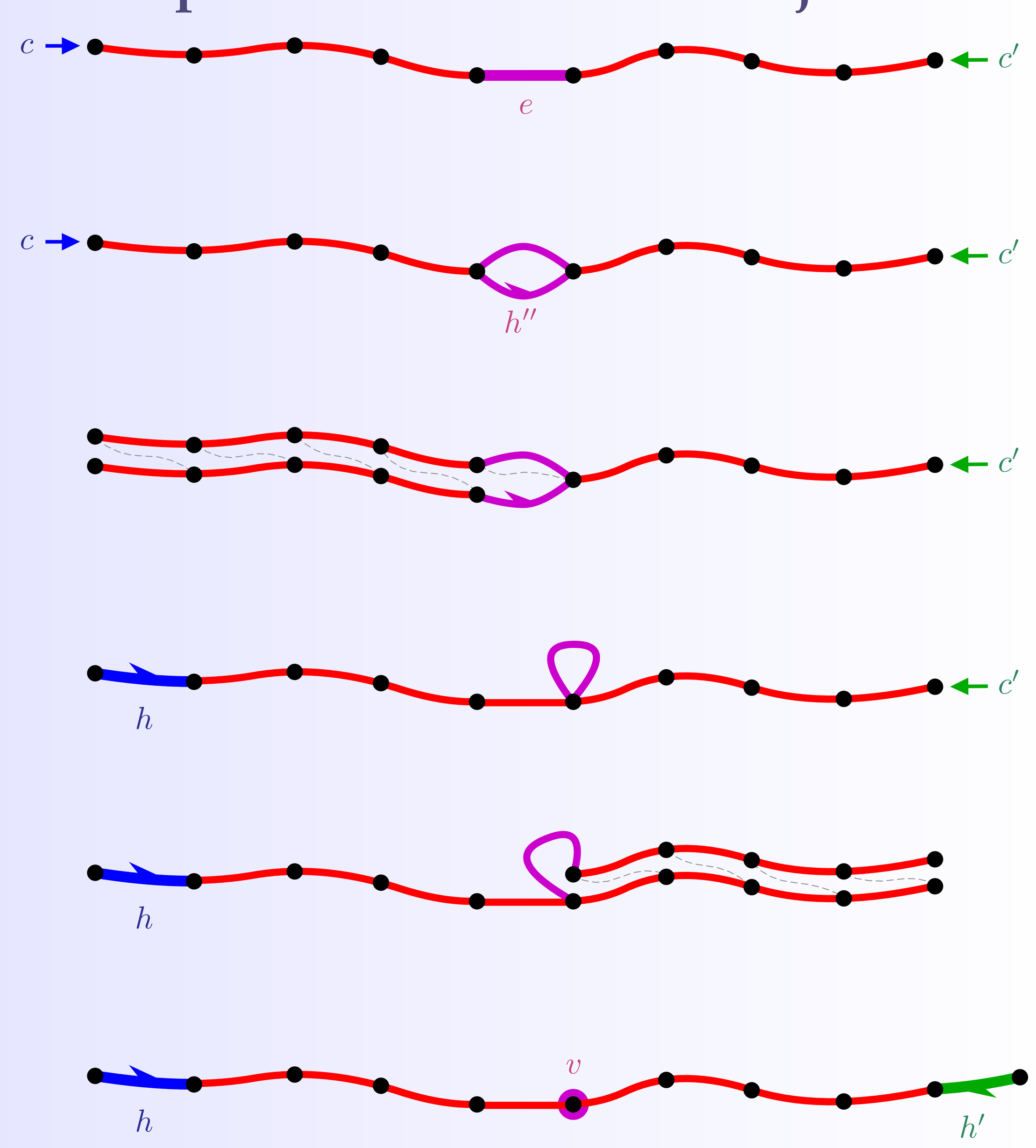
There is thus a circumvention of the image of ℓ in the output map that is either shorter and to the right or strictly shorter and to the left (left part of the picture).

Track back this circumvention in the input map and obtain a circumvention to ℓ that is either shorter and to the left or strictly shorter and to the right (right part of the picture).

This is a contradiction.



Decomposition into transfer bijections



nsup.org/~bettinel

jeremie.bettinelli@normalesup.org