Cluster Algebras and Binary Words

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Binary word

A *binary word* $w = w_1 w_2 \ldots w_d$ is a finite (possibly empty) sequence of letters on the alphabet $\{0,1\}$ starting with 1. A *subword* s of a binary word is a subsequence of w which itself is a binary word.

- Each non-empty binary word can be associated with (the Hasse diagram of) a piece-wise linear poset P.

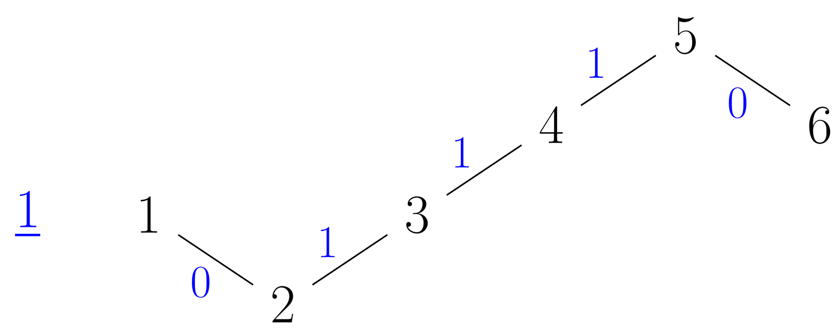


Figure 1. The Hasse diagram associated to the word 101110.

Antichain

An *antichain* is a subset $A = \{A_1, A_2, \ldots, A_r\}$ of a poset such that no two distinct elements in A are comparable.
Example: $\{1,3,6\}, \{1,4\}$ and $\{2,6\}$ are antichains of the poset corresponding to the above Hasse diagram, while $\{2,4\}$ is not.

Trie of subwords (Leroy, Rigo, and Stipulanti)

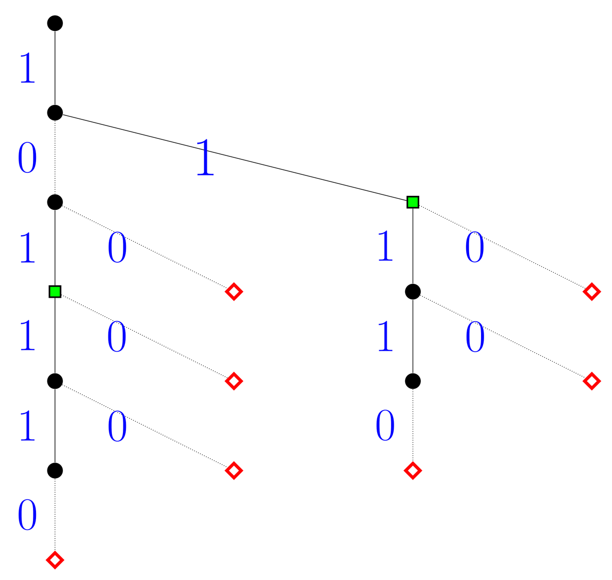


Figure 2. Trie of subwords corresponding to $w = 101110$.

Antichain trie

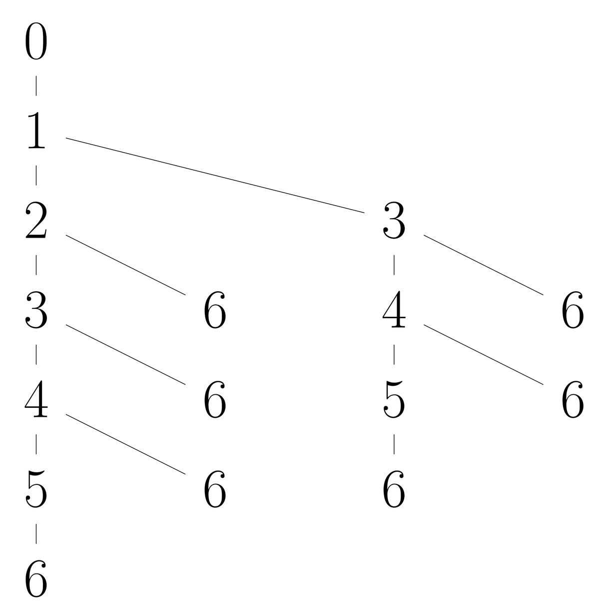


Figure 3. Antichain trie corresponding to $w = 101110$.

Corresponding antichains

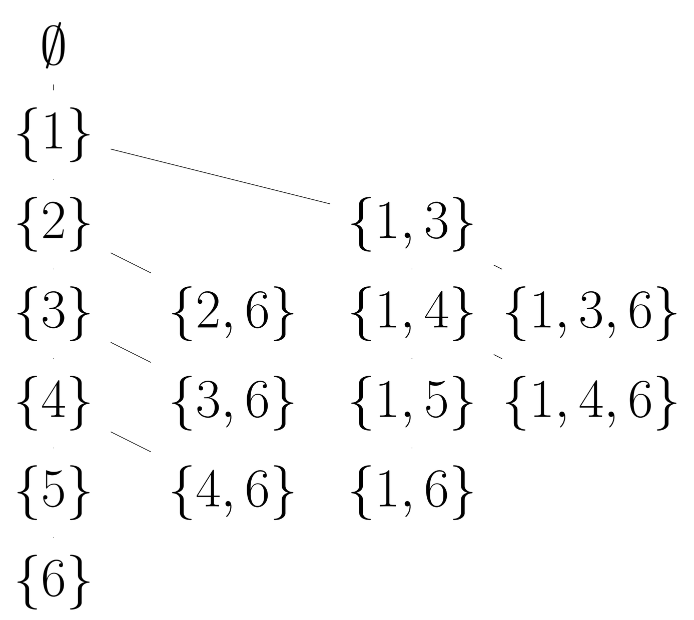
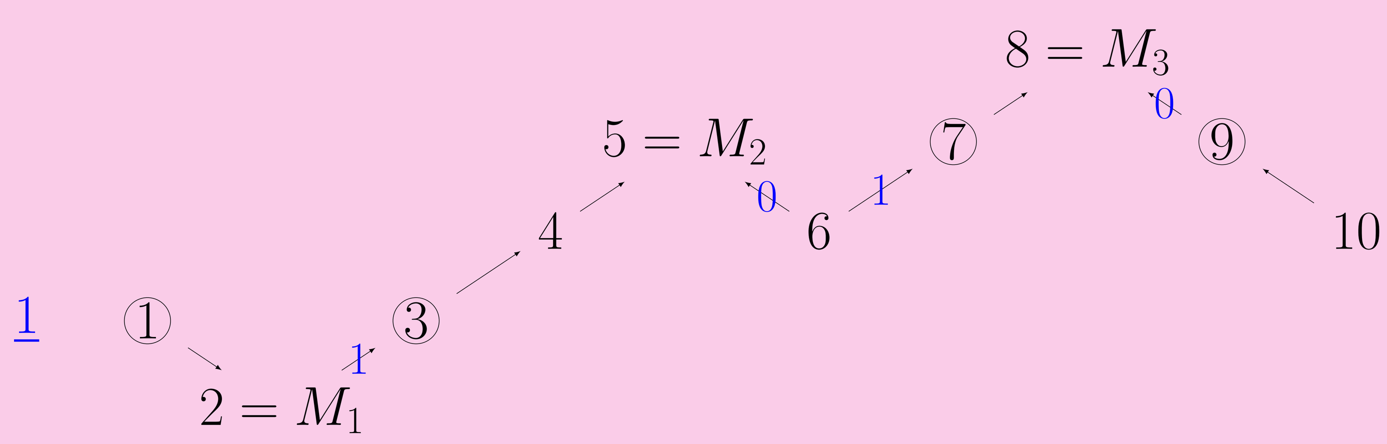


Figure 4. Trie of antichains for $w = 101110$.

Definition

We define a map from the antichains in P to the subwords of w. For example, the antichain A=$\{A_1$=①, A_2=③, A_3=⑦, A_4=⑨$\}$ is mapped to the subword s=1|1|01|0|



Theorem

The map given above is a bijection from the antichains in P to the subwords of w.

Snake graph

A *snake graph* G is a nonempty connected sequence of d square tiles constructed so that each new tile is glued to the north or east side of the previous tile.

- A *perfect matching* of G is a subset of edges of G such that every vertex of G is adjacent to exactly one edge. The *minimal matching* P_{min} is the unique perfect matching of G which contains the first south edge and only boundary edges.

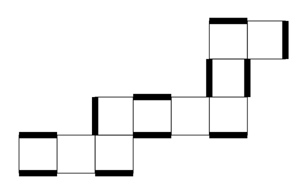


Figure 5. The minimal matching for $w = 1011101100$.

From binary words to snake graphs

- A *cluster algebra* is a commutative algebra with distinguished generators called *cluster variables* which can be written as Laurent polynomials with positive coefficients. Certain Laurent polynomials can be associated to snake graphs.
- A *sign function* on a snake graph G is a map from the set of edges of G to $\{0,1\}$.
- We can associate to each binary word $w = 1w_2 \ldots w_d$ a snake graph $G(w)$.

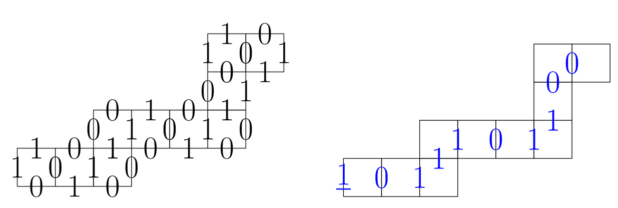


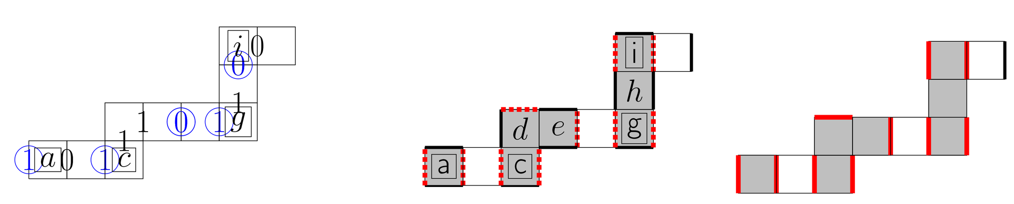
Figure 6. The sign function on the snake graph associated to the word $w = 1011101100$.

Theorem

Given a binary subword w and its corresponding snake graph $G = G(w)$, the following map pm from the subwords of w to the perfect matchings of G is a bijection:

- Let s be a subword of w. If s is the empty word, let $pm(s)$ be P_{min}. Otherwise, write $s = 1w_{i_2} \ldots w_{i_\ell}$ in such a way that each index i_k is as small as possible and circle the edges of G corresponding to the sign sequence for s.
- For each block L of consecutive circled edges, let $\square_L$ be the tile which is immediately north/east of the last edge in L.
- Let $fil(\square_L)$ be the smallest connected sequence of tiles containing $\square_L$ such that the the set of edges bounding $fil(\square_L)$ not in P_{min} forms a perfect matching of $fil(\square_L)$.
- Let $fil(s) = \bigcup_L fil(\square_L)$, and define $pm(s)$ to be the symmetric difference $\{\text{edges bounding } fil(s)\} \ominus P_{\text{min}}$.

Example



The tiles $\square_L$ associated to blocks L of circled edges (left); the set $fil(s)$ of shaded tiles (center) and the set $pm(s)$ of thick solid edges (right) for the subword s=1|1|01|0| of $w = 1011101100$.

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